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Numerical Methods for Partial Differential Equations

ETH Zürich
D-MATH

Homework Problem Sheet 5

Problem 5.1 Linear output functionals

In [NPDE, Section 2.2.3] we have seen that continuity ($\to$ [NPDE, Def. 2.2.49]) of linear forms with respect to energy norms ($\to$ [NPDE, Def. 2.2.38]) induced by symmetric positive definite bilinear forms ($\to$ [NPDE, Def. 2.2.35]) is a key property. Thus, for elliptic boundary value problems, continuity of linear forms in Sobolev spaces is crucial.

For the point evaluation functional, we investigated its continuity in $H^1(\Omega)$ in [NPDE, Ex. 2.4.18], for the source functional $v \to \int_\Omega f v \,\mathrm{d}\boldsymbol{x}$ continuity was studied in [NPDE, Section 2.3.3], whereas boundary functionals arising from non-homogeneous Neumann problems were examined in [NPDE, § 2.10.7].

In this problem we consider the linear functionals

$$J_1(v) := \int_\Omega \mathbf{c} \cdot \mathbf{grad}\, v(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \,, \quad \mathbf{c} \in \mathbb{R}^2 \,, \tag{5.1.1}$$

$$J_2(v) := \int_\Omega v(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \,, \tag{5.1.2}$$

$$J_3(v) := \int_{\partial\Omega} \mathbf{grad}\, v(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x}) \,\mathrm{d}S(\boldsymbol{x}) \,, \tag{5.1.3}$$

$$J_4(v) := \int_\Omega v\left(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\right) \mathrm{d}\boldsymbol{x} \,. \tag{5.1.4}$$

on the unit disk $\Omega = \{\boldsymbol{x} \in \mathbb{R}^2 : \|\boldsymbol{x}\| < 1\}$. These functionals all make sense when we apply them to smooth functions.

Please answer the following questions for (5.1.1)–(5.1.4).

(5.1a) Which of these functionals are continuous on $L^2(\Omega)$? If you suspect a functional to be continuous, try to prove it. If you think, it is not continuous, try to find a counterexample as in [NPDE, § 2.4.20].

HINT: The functional (5.1.4) can be rewritten in terms of an integral over $\partial\Omega$.

Solution: The functional J_1 is *not* continuous on $L^2(\Omega)$.

Consider, for instance, $v(\boldsymbol{x}) = \log(-\log(\sqrt{x_1^2 + x_2^2})) \in L^2(\Omega)$, with $\boldsymbol{x} = (x_1, x_2)$, i.e., in polar coordinates, $\tilde{v}(r,\varphi) = \log(-\log(r))$, $r \in [0,1]$,$\varphi \in [0,2\pi)$. Then

$$\mathbf{grad}\, v(\boldsymbol{x}) = \mathbf{grad}(\tilde{v}(r,\varphi)) = \begin{pmatrix} \dfrac{\partial \tilde{v}}{\partial r}\dfrac{\partial r}{\partial x_1} \\ \dfrac{\partial \tilde{v}}{\partial r}\dfrac{\partial r}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \frac{\cos(\varphi)}{r^2 \log(r)} \\ \frac{\sin(\varphi)}{r^2 \log(r)} \end{pmatrix}$$

and

$$J_1(v) = J_1(\tilde{v}) = \int_0^{2\pi} \int_0^1 \left(c_1 \frac{\cos(\varphi)}{r^2 \log(r)} + c_2 \frac{\sin(\varphi)}{r^2 \log(r)} \right) r \,\mathrm{d}r \,\mathrm{d}\varphi = -\infty.$$

The functional J_2 is continuous on $L^2(\Omega)$. Indeed, using the Cauchy-Schwarz inequality we obtain:

$$\left| \int_\Omega v(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \right| \leq |\Omega| \|v\|_{L^2(\Omega)}, \qquad \text{for all } v \in L^2(\Omega).$$

The functional J_3 is *not* continuous on $L^2(\Omega)$.

To see this, we take the function $v(\boldsymbol{x}) = \log(1 - \|x\|)$. We have that

$$\int_\Omega v^2(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} = 2\pi \int_0^1 \log^2(1-r) r \,\mathrm{d}r < +\infty,$$

and thus $v \in L^2(\Omega)$. However, $\mathbf{grad}\, u = -\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|(1-\|\boldsymbol{x}\|)}$ and

$$\int_{\partial\Omega} \mathbf{grad}\, v(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x}) \,\mathrm{d}S(\boldsymbol{x}) = -\int_{\partial\Omega} \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|(1-\|\boldsymbol{x}\|)} \cdot \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} \,\mathrm{d}S(\boldsymbol{x}) = -\int_{\partial\Omega} \frac{1}{1-\|\boldsymbol{x}\|} = -\infty.$$

The functional J_4 can be rewritten as

$$\begin{aligned} J_4(v) &= \int_\Omega v\left(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\right) \mathrm{d}\boldsymbol{x} = \int_0^1 \int_0^{2\pi} v(\cos\phi, \sin\phi) r \,\mathrm{d}\phi \,\mathrm{d}r = \\ &= \int_0^{2\pi} v(\cos\phi, \sin\phi) \,\mathrm{d}\phi = \int_{\partial\Omega} v(\boldsymbol{x}) \,\mathrm{d}S(\boldsymbol{x}). \end{aligned}$$

If we consider again $v(\boldsymbol{x}) = \log(1 - \|\boldsymbol{x}\|) \in L^2(\Omega)$, then $\int_{\partial\Omega} \log(1-\|\boldsymbol{x}\|) \,\mathrm{d}S(\boldsymbol{x}) = -\infty$, which means that J_4 is *not* continuous on $L^2(\Omega)$.

(5.1b) Solve subproblem (5.1a), now with $L^2(\Omega)$ replaced with the Sobolev space $H^1(\Omega)$.

HINT: The standard tools for proving continuity of linear functionals on Sobolev spaces are the Cauchy-Schwarz inequality [NPDE, Eq. (2.2.39)] and trace theorems like [NPDE, Thm. 2.10.8].

Solution: J_1 is continuous on $H^1(\Omega)$:

$$\begin{aligned} |J_1(v)| &\leq \|\mathbf{c}\|_{\mathbb{R}^2} \int_\Omega \|\mathbf{grad}\, v\|_{\mathbb{R}^2} \,\mathrm{d}\boldsymbol{x} \leq \\ &\leq \|\mathbf{c}\|_{\mathbb{R}^2} |\Omega| |v|_{H^1(\Omega)} \leq \\ &\leq \|\mathbf{c}\|_{\mathbb{R}^2} |\Omega| \|v\|_{H^1(\Omega)}, \qquad \text{for all } v \in H^1(\Omega), \end{aligned}$$

where in the first and second step we have used Cauchy-Schwarz inequality.

Also J_2 is continuous on $H^1(\Omega)$. Indeed, using Cauchy-Schwarz inequality we obtain:

$$|J_2(v)| = \left| \int_\Omega v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \right| \leq |\Omega| \|v\|_{L^2(\Omega)} \leq |\Omega| \|v\|_{H^1(\Omega)}, \quad \text{for all } v \in H^1(\Omega).$$

J_3 is *not* continuous on $H^1(\Omega)$. For example, if we take $v(\boldsymbol{x}) = (1 - \|\boldsymbol{x}\|)\log(1 - \|\boldsymbol{x}\|)$, then $\mathbf{grad}\, v = -\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}(\log(1 - \|\boldsymbol{x}\| + 1))$ and therefore

$$\int_{\partial\Omega} \mathbf{grad}\, v(\boldsymbol{x}) \cdot \boldsymbol{x}(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{x}) = -\int_{\partial\Omega} \log(1 - \|x\|) + 1\,\mathrm{d}S(\boldsymbol{x}) = +\infty.$$

J_4 is continuous on $H^1(\Omega)$:

$$\begin{aligned}
\left| \int_\Omega v\left(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\right) \mathrm{d}\boldsymbol{x} \right| &= \left| \int_{\partial\Omega} v(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{x}) \right| \leq \\
&\leq |\partial\Omega| \|v\|_{L^2(\Omega)} \leq \\
&\leq |\partial\Omega| C(\Omega) \sqrt{\|v\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)}} \leq \\
&\leq |\partial\Omega| C(\Omega) \|v\|_{H^1(\Omega)},
\end{aligned}$$

for all $v \in H^1(\Omega)$, where for the first inequality we have used Cauchy-Schwarz inequality, and for the second one we have used the multiplicative trace inequality ([NPDE, Thm. 2.10.8]).

Problem 5.2 Heat Conduction with Non-Local Boundary Conditions

This problem is meant to practice the conversion of a variational problem into a boundary value for a partial differential equation, see [NPDE, Section 2.5] and the extraction of boundary conditions hidden in the variational formulation as in [NPDE, Ex. 2.5.18].

Concretely, we consider the modelling of a two-dimensional cross-section of a submerged insulated wire, see Figure 5.1. The wire has a central core of conducting material, say copper, which carries a current. Ohmic losses lead to heat generation in the copper. Copper conducts heat very well and, thus, the copper core can be assumed to have a *uniform but unknown* temperature.

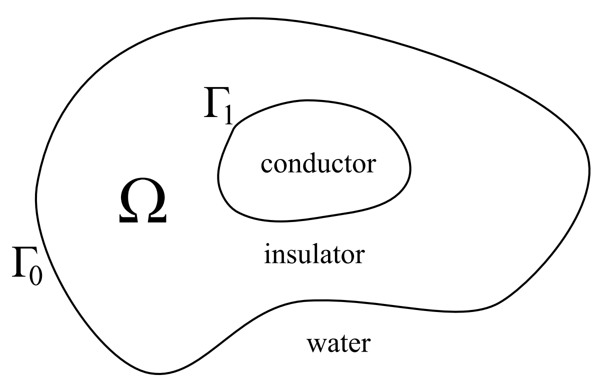

Figure 5.1: Cross-section of a submerged wire.

The copper is surrounded by an annulus of insulator, some plastic, for example, which is again surrounded by water, which we assume to be at a constant temperature of 0. We seek a mathematical model providing us with the temperature distribution within the insulation. Such a model is given by the variational problem

$$u \in V_0 : \int_\Omega \kappa(\mathbf{x}) \operatorname{grad} u(\mathbf{x}) \cdot \operatorname{grad} v(\mathbf{x}) \,\mathrm{d}\mathbf{x} = \int_{\Gamma_1} v(\mathbf{x}) \,\mathrm{d}S\,, \quad \forall v \in V_0, \tag{5.2.1}$$

where the heat conductivity κ is uniformly positive ($\rightarrow$ [NPDE, Def. 2.2.15]) and bounded, and with

$$V_0 = \left\{ v \in H^1(\Omega) \,\middle|\, v|_{\Gamma_0} = 0,\, v|_{\Gamma_1} = \text{const} \right\}.$$

(5.2a) Determine a bilinear form a and linear form ℓ so that (5.2.1) becomes an abstract linear variational problem $\mathsf{a}(u, v) = \ell(v)$.

Solution: We have

$$\mathsf{a}(u, v) = \int_\Omega \kappa(\mathbf{x}) \operatorname{grad} u(\mathbf{x}) \cdot \operatorname{grad} v(\mathbf{x}) \,\mathrm{d}\mathbf{x},$$

and

$$\ell(v) = \int_{\Gamma_1} v(\mathbf{x}) \,\mathrm{d}S.$$

(5.2b) Show that ℓ is continuous with respect to the energy norm induced by a, *cf.* [NPDE, Def. 2.2.49]. In the lecture we found this to be an essential condition for the well-posedness of a linear variational problem, see [NPDE, Lemma 2.2.47].

HINT: The energy norm is defined as in [NPDE, Def. 2.2.38], and ℓ must satisfy [NPDE, Eq. (2.2.48)] to be continuous with respect to this norm. Then use the *trace theorem* [NPDE, Thm. 2.10.8].

Solution: Using Cauchy-Schwarz, [NPDE, Thm. 2.10.8], Triangle inequality and [NPDE, Thm. 2.3.16],

$$\begin{aligned}
|\ell(\mathbf{v})|^2 &= \left| \int_{\Gamma_1} v(\mathbf{x}) \,\mathrm{d}\mathbf{x} \right|^2 \leq \left(\int_{\Gamma_1} \mathrm{d}\mathbf{x} \right) \left(\int_{\Gamma_1} |v(\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} \right) \leq C_0 \|v\|^2_{L^2(\Gamma_1)} \\
&= C_0 \|v\|^2_{L^2(\partial\Omega)} \leq C_1 \|v\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)} \leq C_1 \|v\|_{L^2(\Omega)} \left(\|v\|_{L^2(\Omega)} + |v|_{H^1(\Omega)} \right) \\
&\leq C_2 |v|_{H^1(\Omega)} \left(C_3 |v|_{H^1(\Omega)} + |v|_{H^1(\Omega)} \right) \leq C_4 |v|^2_{H^1(\Omega)} \leq \frac{C_4}{\underline{\kappa}} \|v\|^2_{\mathsf{a}},
\end{aligned}$$

which concludes the proof. Here, $\underline{\kappa}$ is a lower bound for $\kappa(\mathbf{x})$.

(5.2c) If u solves (5.2.1) and is sufficiently smooth, it also satisfies a partial differential equation on Ω. Find this equation.

HINT: Follow the approach of [NPDE, Section 2.5]: as test functions v use functions in $C^1_0(\Omega)$, that is, they should be zero on both boundaries Γ_0, Γ_1. Use [NPDE, Thm. 2.5.9] (with $\operatorname{grad} u$ in place of j). Argue what happens to the boundary terms. Then appeal to [NPDE, Lemma 2.5.12].

Solution: For $v \in C^1(\Omega)$ we have

$$\begin{aligned}
\int_\Omega \kappa(\mathbf{x}) \operatorname{grad} u(\mathbf{x}) \cdot \operatorname{grad} v(\mathbf{x}) \,\mathrm{d}\mathbf{x} &= \int_{\partial\Omega} \kappa(\mathbf{x}) v(\mathbf{x}) \operatorname{grad} u(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) \,\mathrm{d}\mathbf{x} \\
&\quad - \int_\Omega v(\mathbf{x}) \operatorname{div} \kappa(\mathbf{x}) \operatorname{grad} u(\mathbf{x}) \,\mathrm{d}\mathbf{x}.
\end{aligned} \tag{5.2.2}$$

Now if we further restrict ourselves to $v \in C_0^1(\Omega)$, then $v = 0$ on $\partial\Omega$. Using (5.2.2), we reduce (5.2.1) to

$$-\int_\Omega v(\mathbf{x}) \operatorname{div} \kappa(\mathbf{x}) \operatorname{grad} u(\mathbf{x}) \, \mathrm{d}\mathbf{x} = 0.$$

This holds for every $v \in C_0^1(\Omega)$ and [NPDE, Lemma 2.5.12] implies $\operatorname{div} \kappa(\mathbf{x}) \operatorname{grad} u(\mathbf{x}) = 0$ in Ω. Note that if $\kappa = \text{const}$, then this becomes $-\Delta u(\mathbf{x}) = 0$,the familiar Laplacian.

(5.2d) The function u from problem (5.2c) must also satisfy a certain non-local boundary condition implied by (5.2.1). Find this boundary condition.

HINT: Follow the strategy from [NPDE, Ex. 2.5.18] and use the PDE derived in the previous sub-problem.

Solution: First we notice that $u \in V_0$ means that we have a homogeneous Dirichlet boundary condition on Γ_0 (natural boundary condition). We still need to figure out the boundary condition on Γ_1. For $v \in V_0$ in (5.2.1) we first use (5.2.2) and then use the PDE derived in the previous subtask to cancel all terms inside the domain, the only remaining terms are then

$$\int_{\partial\Omega} \kappa(\mathbf{x}) v(\mathbf{x}) \operatorname{grad} u(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) \, \mathrm{d}\mathbf{x} = \int_{\Gamma_1} v(\mathbf{x}) \, \mathrm{d}\mathbf{x}.$$

We note that the integral over Γ_0 will disappear because $v = 0$ there. On Γ_1, we must have $v = \text{const}$, which only shows that

$$\int_{\Gamma_1} \kappa(\mathbf{x}) \operatorname{grad} u(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) \, \mathrm{d}\mathbf{x} = \int_{\Gamma_1} 1 \, \mathrm{d}\mathbf{x} = |\Gamma_1|.$$

So, this boundary condition is non-local.

(5.2e) What is the physical interpretation of the boundary condition from (5.2d) in terms of heat conduction?

Solution: It specifies the total heat flux over the boundary Γ_1.

Problem 5.3 Minimization of a Quadratic Functional

[NPDE, Section 2.2.3] introduced abstract quadratic minimization problems, see [NPDE, Def. 2.2.24] and [NPDE, Def. 2.2.29]. As concrete examples arising from equilibrium models we studied quadratic minimization problems posed on the Sobolev spaces $H_0^1(\Omega)$ and $H^1(\Omega)$ of scalar functions. In [NPDE, Section 2.4], we learned how to convert a quadratic minimization problem into variational form, see [NPDE, Eq. (2.4.9)]. [NPDE, Section 2.5] taught us how to use multidimensional integration by parts [NPDE, Thm. 2.5.9] to convert the linear variational problems on Sobolev spaces into a boundary value problems for 2nd-order elliptic PDEs. In this exercise we practise all these steps in the case of an “exotic” quadratic minimization problem.

We consider the quadratic functional

$$J(\mathbf{u}) = \int_\Omega \left| \operatorname{div} \mathbf{u}(\mathbf{x}) \right|^2 + \|\mathbf{u}(\mathbf{x})\|^2 + \mathbf{f}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) \, \mathrm{d}\mathbf{x} \,, \tag{5.3.1}$$

with $\Omega \subset \mathbb{R}^3$ bounded, and for functions $\mathbf{u} : \Omega \to \mathbb{R}^3$, that is, J takes *vector field arguments*.

(5.3a) Identify the bilinear form a and the linear form ℓ in the quadratic functional J, *cf.* [NPDE, Def. 2.2.24].

HINT: See [NPDE, Def. 2.2.24].

Solution: We get

$$\mathsf{a}(\mathbf{u},\mathbf{v}) = 2\int_\Omega \operatorname{div}\mathbf{u}(\mathbf{x})\operatorname{div}\mathbf{v}(\mathbf{x})\,\mathrm{d}\mathbf{x} + 2\int_\Omega \mathbf{u}(\mathbf{x})\cdot\mathbf{v}(\mathbf{x})\,\mathrm{d}\mathbf{x},$$

and

$$\ell(\mathbf{v}) = -\int_\Omega \mathbf{f}(\mathbf{x})\cdot\mathbf{v}(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

(5.3b) Show that the bilinear form a from subproblem (5.3a) is symmetric and positive definite, see [NPDE, Def. 2.2.35].

HINT: See [NPDE, Eq. (2.2.26)] and [NPDE, Def. 2.2.35].

Solution: a is clearly symmetric. To show that it is positive definite, assume

$$0 = \mathsf{a}(\mathbf{u},\mathbf{u}) = 2\int_\Omega \left|\operatorname{div}\mathbf{u}(\mathbf{x})\right|^2\mathrm{d}\mathbf{x} + 2\int_\Omega \left\|\mathbf{u}(\mathbf{x})\right\|^2\mathrm{d}\mathbf{x}.$$

Since both integrands are nonnegative, we must have $\mathbf{u}(\mathbf{x}) = 0$ almost everywhere in Ω, i.e., $\mathbf{u} = 0$. This shows that $\mathsf{a}(\mathbf{u},\mathbf{u}) > 0$ whenever $\mathbf{u} \neq 0$.

(5.3c) Show that the linear form ℓ from subproblem (5.3a) is continuous with respect to the energy norm induced by a.

HINT: The energy norm is defined as in [NPDE, Def. 2.2.38], and ℓ must satisfy [NPDE, Eq. (2.2.48)] to be continuous with respect to this norm.

Solution: Once more, the Cauchy-Schwartz inequality comes to our rescue.

$$\begin{aligned}
|\ell(\mathbf{v})| &= \left|-\int_\Omega \mathbf{f}(\mathbf{x})\cdot\mathbf{v}(\mathbf{x})\,\mathrm{d}\mathbf{x}\right| \\
&\leq \left(\int_\Omega \left\|\mathbf{f}(\mathbf{x})\right\|^2\mathrm{d}\mathbf{x}\right)^{\frac{1}{2}}\left(\int_\Omega \left\|\mathbf{v}(\mathbf{x})\right\|^2\mathrm{d}\mathbf{x}\right)^{\frac{1}{2}} \\
&\leq \left(\int_\Omega \left\|\mathbf{f}(\mathbf{x})\right\|^2\mathrm{d}\mathbf{x}\right)^{\frac{1}{2}}\left(\int_\Omega \left\|\mathbf{v}(\mathbf{x})\right\|^2\mathrm{d}\mathbf{x} + \int_\Omega \left\|\operatorname{div}\mathbf{v}(x)\right\|^2\mathrm{d}\mathbf{x}\right)^{\frac{1}{2}} \\
&= \left(\int_\Omega \left\|\mathbf{f}(\mathbf{x})\right\|^2\mathrm{d}\mathbf{x}\right)^{\frac{1}{2}}\sqrt{\mathsf{a}(\mathbf{v},\mathbf{v})}
\end{aligned}$$

(5.3d) Explain why the Sobolev space

$$H(\operatorname{div},\Omega) := \left\{\mathbf{v}:\Omega\to\mathbb{R}^3 \text{ integrable} \;\middle|\; \int_\Omega \left|\operatorname{div}\,\mathbf{v}\right|^2 + \left\|\mathbf{v}\right\|^2\mathrm{d}\mathbf{x} < \infty\right\}.$$

provides the right framework for studying the minimization problem for the functional J from (5.3.1).

Solution: This is exactly the space

$$\left\{\mathbf{v}:\Omega\to\mathbb{R}^3 \text{ integrable} \;\middle|\; \left\|\mathbf{v}\right\|_{\mathsf{a}} < \infty\right\}.$$

(5.3e) Derive and state the linear variational problem equivalent to the minimization problem

$$\mathbf{u}_* = \underset{\mathbf{v}\in H(\mathrm{div},\Omega)}{\operatorname{argmin}} \; J(\mathbf{v}).$$

HINT: See [NPDE, Eq. (2.4.8)] and [NPDE, Eq. (2.4.9)].

Solution: Find $\mathbf{u} \in H(\mathrm{div}, \Omega)$ such that

$$\mathsf{a}(\mathbf{u}, \mathbf{v}) = 2\int_{\Omega} \mathrm{div}\,\mathbf{u}(\mathbf{x})\,\mathrm{div}\,\mathbf{v}(\mathbf{x})\,\mathrm{d}\mathbf{x} + 2\int_{\Omega} \mathbf{u}(\mathbf{x})\cdot\mathbf{v}(\mathbf{x})\,\mathrm{d}\mathbf{x} = -\int_{\Omega} \mathbf{f}(\mathbf{x})\cdot\mathbf{v}(\mathbf{x})\,\mathrm{d}\mathbf{x} = \ell(\mathbf{v}) \quad (5.3.2)$$

for all $\mathbf{v} \in H(\mathrm{div}, \Omega)$.

(5.3f) Derive the *partial differential equation* on Ω that arises from the variational problem from (5.3e).

HINT: Follow the approach of [NPDE, Section 2.5], in particular [NPDE, Ex. 2.5.18]: as test functions $\mathbf{v}$ use vector fields in $(\mathcal{C}_0^1(\Omega))^3$, that is, they should be zero on the boundary. Use [NPDE, Thm. 2.5.9] (with $\mathrm{div}\ \mathbf{u}$ in place of v and $\mathbf{v}$ in place of $\mathbf{j}$) in order to "shift the div from $\mathbf{v}$ onto $\mathrm{div}\,u$ as $-\,\mathrm{grad}$". Argue, what happens to the boundary terms. Then appeal to [NPDE, Lemma 2.5.12].

Solution: After using [NPDE, Thm. 2.5.9] on (5.3.2), we get

$$2\int_{\partial\Omega} \mathbf{v}(\mathbf{x})\cdot\mathbf{n}(\mathbf{x})\ \mathrm{div}\,\mathbf{u}(\mathbf{x})\,\mathrm{d}\mathbf{x} + \int_{\Omega} \mathbf{v}(\mathbf{x})\cdot[2\mathbf{u}(\mathbf{x}) - 2\,\mathrm{grad}(\mathrm{div}\,\mathbf{u}(\mathbf{x})) + \mathbf{f}(\mathbf{x})]\,\mathrm{d}\mathbf{x} = 0. \quad (5.3.3)$$

By choosing $\mathbf{v} = 0$ on $\partial\Omega$, the boundary term will disappear, and since $\mathbf{v}$ is otherwise arbitrary, we obtain

$$2\mathbf{u}(\mathbf{x}) - 2\,\mathrm{grad}(\mathrm{div}\,\mathbf{u}(\mathbf{x})) = -\mathbf{f}(\mathbf{x})$$

in Ω.

(5.3g) The variational problem from (5.3e) also implies boundary conditions. Which?

HINT: Follow the strategy from [NPDE, Ex. 2.5.18] and use the PDE derived in subproblem (5.3f).

Solution: According to what we found in the last problem, (5.3.3) reduces to

$$\int_{\partial\Omega} 2\mathbf{v}(\mathbf{x})\cdot\mathbf{n}(\mathbf{x})\ \mathrm{div}\,\mathbf{u}(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0,$$

from which we obtain the boundary condition

$$\mathrm{div}\,\mathbf{u}(\mathbf{x}) = 0$$

on $\partial\Omega$.

Problem 5.4 Poisson equation in polar coordinates

In the problem we will come across an important case of *transformation* of the domain of a boundary value problem prior to its discretization. We interpret the domain transformation as a *change of coordinates*, studying the concrete case of *polar coordinates* to switch from the unit disk domain to a simple square domain.

Remark. A rationale for using polar coordinates when dealing with boundary value problems on a disk is that, of course, mesh generation is trivial for a square and boundary approximation is not a concern. This will become in [NPDE, Chapter 3].

As a model problem we consider homogeneous Dirichlet problem for the Poisson equation [NPDE, Eq. (2.5.15)]

$$-\Delta u = f \quad \text{in}\,\Omega\,, \qquad u = 0 \quad \text{on}\,\partial\Omega\,, \tag{5.4.1}$$

on the unit disk

$$\Omega = \{\boldsymbol{x} \in \mathbb{R}^2 : |\boldsymbol{x}| < 1\}\,. \tag{5.4.2}$$

Here Δ is the Laplace operator, see [NPDE, Rem. 2.5.14]. The variational (weak) formulation of (5.4.1) has been discussed in [NPDE, Ex. 2.9.2].

The transformation from polar coordinates (r, ϕ), $0 \leq r \leq 1$, $0 \leq \phi < 2\pi$, to Cartesian coordinates $(x_1, x_2) \in \mathbb{R}^2$ is given by the mapping

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \boldsymbol{\Phi}(r, \phi) := r \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \tag{5.4.3}$$

(*cf.* [NPDE, Eq. (2.4.21)]), and we have $\Omega = \boldsymbol{\Phi}(\Omega_p)$, with the tensor product domain

$$\Omega_p := [0, 1] \times [0, 2\pi]. \tag{5.4.4}$$

Before you start solving this problem, we suggest you to refresh your knowledge about the polar coordinate example in [NPDE, § 2.4.20].

(5.4a) For a function $u \in C^1(\bar{\Omega})$. Compute the *Cartesian* components of $\mathbf{grad}\, u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\right)^T$, for $u = u(r, \phi)$ given in *polar* coordinates, in terms of the partial derivatives $\dfrac{\partial u}{\partial r}$ and $\dfrac{\partial u}{\partial \phi}$.

HINT: Use the chain rule for differentiation.

Solution: We have:

$$\begin{aligned} \frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial r} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial r} = \cos\phi\frac{\partial u}{\partial x_1} + \sin\phi\frac{\partial u}{\partial x_2} \\ \frac{\partial u}{\partial \phi} &= \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial \phi} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial \phi} = -r\sin\phi\frac{\partial u}{\partial x_1} + r\cos\phi\frac{\partial u}{\partial x_2}. \end{aligned}$$

In matrix-vector notation, we can write:

$$\begin{bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial u}{\partial \phi} \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\phi & \sin\phi \\ -r\sin\phi & +r\cos\phi \end{bmatrix}}_{:=D\boldsymbol{\Phi}^T} \begin{bmatrix} \dfrac{\partial u}{\partial x_1} \\ \dfrac{\partial u}{\partial x_2} \end{bmatrix}, \tag{5.4.5}$$

and thus

$$\begin{bmatrix} \dfrac{\partial u}{\partial x_1} \\ \dfrac{\partial u}{\partial x_2} \end{bmatrix} = \underbrace{\frac{1}{r}\begin{bmatrix} r\cos\phi & -\sin\phi \\ r\sin\phi & +\cos\phi \end{bmatrix}}_{:=D\boldsymbol{\Phi}^{-T}} \begin{bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial u}{\partial \phi} \end{bmatrix}. \tag{5.4.6}$$

(5.4b) Explain the origin of the r-factor in the integration formula in polar coordinates:

$$\int_\Omega u(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_0^1 \int_0^{2\pi} u(r,\phi) r \,\mathrm{d}\phi\,\mathrm{d}r. \tag{5.4.7}$$

HINT: You may appeal to the transformation formula for multi-dimensional integrals that you learned in your Analysis course.

Solution: According to the transformation formula for multi-dimensional integrals, it holds that $\mathrm{d}\boldsymbol{x} = \det D\boldsymbol{\Phi}\,\mathrm{d}\phi\,\mathrm{d}r$. From (5.4.6), it follows that $\det D\boldsymbol{\Phi} = r\cos^2\phi + r\sin^2\phi = r$, which explains the factor in (5.4.7).

Alternative: geometric argument, area of angular sectors grows linearly with r.

(5.4c) As we learned in [NPDE, Section 2.9], the bilinear form associated with the homogeneous Dirichlet problem for the linear scalar 2nd-order differential operator $-\Delta$ on Ω reads:

$$\mathsf{a}(u,v) = \int_\Omega \mathbf{grad}\, u(\boldsymbol{x}) \cdot \mathbf{grad}\, v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\ , \quad u,v \in H_0^1(\Omega)\ . \tag{5.4.8}$$

Rewrite it in terms of polar coordinates, that is, for $u = u(r,\phi)$ and $v = v(r,\phi)$, in terms of partial derivatives with respect to r and ϕ, and by means of an integral over the domain Ω_p as given in (5.4.4).

Solution: Let $\mathbf{grad}_p = \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \phi}\right)^T$.

With the help of (5.4.6) and (5.4.7), we have:

$$\begin{aligned}
\int_\Omega \mathbf{grad}\, u \cdot \mathbf{grad}\, v\,\mathrm{d}\boldsymbol{x} &= \int_0^1\int_0^{2\pi} (D\boldsymbol{\Phi}^{-T}\,\mathbf{grad}_p\, u)\cdot(D\boldsymbol{\Phi}^{-T}\,\mathbf{grad}_p\, v)\det D\boldsymbol{\Phi}\,\mathrm{d}\phi\,\mathrm{d}r = \\
&= \int_0^1\int_0^{2\pi} \begin{bmatrix} \frac{\partial u}{\partial r}\cos\phi - \frac{1}{r}\frac{\partial u}{\partial \phi}\sin\phi \\ \frac{\partial u}{\partial r}\sin\phi + \frac{1}{r}\frac{\partial u}{\partial \phi}\cos\phi \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial v}{\partial r}\cos\phi - \frac{1}{r}\frac{\partial v}{\partial \phi}\sin\phi \\ \frac{\partial v}{\partial r}\sin\phi + \frac{1}{r}\frac{\partial v}{\partial \phi}\cos\phi \end{bmatrix} r\,\mathrm{d}\phi\,\mathrm{d}r = \\
&= \int_0^1\int_0^{2\pi} \left(\frac{\partial u}{\partial r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial u}{\partial \phi}\frac{\partial v}{\partial \phi}\right) r\,\mathrm{d}\phi\,\mathrm{d}r = \\
&= \int_0^1\int_0^{2\pi} r\frac{\partial u}{\partial r}\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \phi}\frac{\partial v}{\partial \phi}\,\mathrm{d}\phi\,\mathrm{d}r
\end{aligned} \tag{5.4.9}$$

(5.4d) Let Ω_p be as in (5.4.4). Assuming that $u_p \in C^1(\bar{\Omega}_p)$, what further condition does u_p have to satisfy in order to ensure that $|u|_{H^1(\Omega)} < \infty$, where $u(x_1,x_2) := u_p(r(x_1,x_2),\phi(x_1,x_2)) : \Omega \to \mathbb{R}$ (and (r,ϕ) are the polar coordinates on Ω as given in (5.4.3))?

HINT: Use the results from subproblem (5.4c).

Solution: Using (5.4.9):

$$|u|^2_{H^1(\Omega)} = \int_\Omega \mathbf{grad}\, u(\boldsymbol{x})\cdot\mathbf{grad}\, u(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_{\Omega_p} r\left(\frac{\partial u_p}{\partial r}(r,\phi)\right)^2 + \frac{1}{r}\left(\frac{\partial u_p}{\partial \phi}(r,\phi)\right)^2 \mathrm{d}\phi\,\mathrm{d}r.$$

In the second addend of the last integral, we have the factor $\frac{1}{r}$, which is not integrable at $r = 0$. In order to have a finite integral, we need to integrate a quantity that behaves as $\frac{1}{r^\alpha}$, with $\alpha < 1$.

This means that we need $\left(\frac{\partial u_p}{\partial \phi}(r,\phi)\right)^2 = O(r^\beta)$ for $r \to 0$, with $\beta > 0$. In other words, we need that $\frac{\partial u_p}{\partial \phi}(0,\phi) = 0$.

Write $u \in H_0^1(\Omega)$ for the weak solution on (5.4.1), and $u_p : \Omega_p \to \mathbb{R}$ for its transformation into polar coordinates: $u_p(r,\phi) := u(x_1(r,\phi), x_2(r,\phi))$.

(5.4e) What linear variational problem on Ω_p is solved by u_p? Assume that also f is given in polar coordinates: $f = f(r,\phi)$.

HINT: The results from task (5.4c) may come handy.

Solution: The variational formulation on Ω_p reads:

Find $u_p \in V_{0,p} := \left\{ v_p \in H^1(\Omega_p) : \frac{\partial v_p}{\partial \phi}(0,\phi) = 0 \text{ and } v_p(1,\phi) = 0 \text{ for all } \phi \in [0, 2\pi) \right\}$ such that

$$\underbrace{\int_{\Omega_p} r\frac{\partial u_p}{\partial r}(r,\phi)\frac{\partial v_p}{\partial r}(r,\phi) + \frac{1}{r}\frac{\partial u_p}{\partial \phi}(r,\phi)\frac{\partial v_p}{\partial \phi}(r,\phi)\,\mathrm{d}\phi\,\mathrm{d}r}_{\mathsf{a}_p(u_p,v_p)} = \underbrace{\int_{\Omega_p} f(r,\phi)v_p(r,\phi)r\,\mathrm{d}\phi\,\mathrm{d}r}_{\ell(v_p)} \quad (5.4.10)$$

for all $v_p \in V_{0,p}$.

Now we assume that the source function enjoys rotational symmetry, i.e. $f = f(r)$, with no dependence on ϕ. Then the solution to (5.4.1) will also be rotationally symmetric: $u_p = u_p(r)$, $0 \le r \le 1$.

(5.4f) What variational problem (in polar coordinates) has to be satisfied by the rotationally symmetric solution $u_p = u_p(r)$ of (5.4.1) in the case of $f = f(r)$?

Solution: Since for the test functions too it would not make sense not to choose them to be rotationally symmetric, the variational formulation reads:

Find $u_p \in W := \left\{ v \in \mathcal{C}^1_{pw}([0,1]) : v(1) = 0 \right\}$ such that:

$$\int_0^1 r\frac{du_p}{dr}(r)\frac{dv}{dr}(r)\,\mathrm{d}r = \int_0^1 f(r)v(r)r\,\mathrm{d}r \quad \text{for all } v \in W. \quad (5.4.11)$$

(5.4g) The *energy space* for the variational problem from task (5.4f) is:

$$V := \left\{ v \in L^2(]0,1[) : \int_0^1 r\left|\frac{dv}{dr}(r)\right|^2 \mathrm{d}r < \infty,\ v(1) = 0 \right\}. \quad (5.4.12)$$

Is the linear functional $J : V \to \mathbb{R}$ given by the point evaluation $J(v) = v(0)$ continuous on V?

HINT: Follow the approach of [NPDE, § 2.4.20] and try to find a function $v \in V$ with "$v(0) = \infty$". It is worth studying [NPDE, § 2.4.20] carefully, because after transformation back to the disk Ω, V can be regarded as the space of rotationally symmetric functions in $H_0^1(\Omega)$.

Solution: No, it is not. As seen in [NPDE, § 2.4.20], let us consider $v = \log|\log \frac{r}{e}|$, $r \neq 0$.

Then we have that $v \in V$ (see [NPDE, § 2.4.20]) but $J(v) = v(0) = \infty$.

(5.4h) Assuming that $u_p \in \mathcal{C}^2([0,1])$, state the 2-point boundary value problem associated to the variational formulation from task (5.4f).

HINT: The boundary conditions will look strange, but, in light of the discussion in [NPDE, Rem. 2.3.6], the result of subproblem (5.4g) should make clear, why imposing boundary values at 0 does not make sense.

Solution: Integration by parts of (5.4.11) gives:

$$-\int_0^1 \frac{\mathrm{d}}{dr}\left(r\frac{du_p}{dr}\right)(r)v(r)\,\mathrm{d}r + \left[r\frac{du_p}{dr}(r)v(r)\right]_0^1 = \int_0^1 rf(r)v(r)\,\mathrm{d}r.$$

For density argument, we can take v such that $v(0) = v(1) = 0$. Then the 2-point boundary value problem reads:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}u_p(r)\right) &= rf(r) \quad \text{in }]0,1[, \\ u(1) &= 0. \end{aligned}$$

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