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Numerical Methods for Partial Differential Equations

ETH Zürich
D-MATH

Homework Problem Sheet 9

Problem 9.1 Midpoint Quadrature on Triangles (Core problem)

Local quadrature is a core operation in any finite element computation, see [NPDE, Section 3.6.4]. This exercise examines a special local quadrature rule, the concept of order and the use of local quadrature for computing element matrices.

The midpoint quadrature rule on the unit triangle is given by

$$\int_{\widehat{K}} f(\widehat{\mathbf{x}})\,\mathrm{d}\widehat{\mathbf{x}} \approx \frac{1}{6}\sum_i f(\mathbf{m}^i),$$

where $\mathbf{m}^i$ are the midpoints of the edges of $\widehat{K}$.

(9.1a) Show that the midpoint quadrature rule is exact for $f \in \mathcal{P}_2(\mathbb{R}^2)$ that is, it is of *order* 3.

HINT: Find a simple basis for $f \in \mathcal{P}_2(\mathbb{R}^2)$ (see [NPDE, Def. 3.4.8]), integrate each basis function and then compare to the value the quadrature rule would give. Since integration is linear, this result will extend to all of $\mathcal{P}_2(\mathbb{R}^2)$. You may use [NPDE, Lemma 3.6.61].

(9.1b) Let K be any triangle. Find the element matrix for the bilinear form

$$\mathsf{a}_K(u,v) = \int_K u(\mathbf{x})v(\mathbf{x})\,\mathrm{d}\mathbf{x}$$

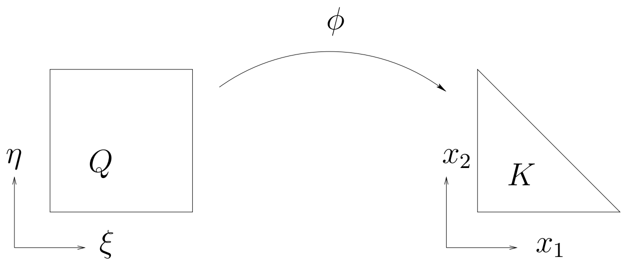
using midpoint quadrature, for Lagrangian finite elements of local polynomial degree 1 and 2.

HINT: See [NPDE, Section 3.3.5] and [NPDE, Eq. (3.5.6)] for the basis functions in barycentric coordinates. You can use [NPDE, Lemma 3.6.61]. Assume that the area $|K|$ of the triangle is known.

Problem 9.2 The Duffy Trick

In [NPDE, Section 3.6.4.2] we learned that local quadrature rules on the cells of a finite element mesh can be obtained by transformation, see [NPDE, § 3.6.83] for details. In this problem we witness the use of a special transformation to generate quadrature rules on the reference triangle.

Consider the following mapping from the unit square $Q := (0,1)^2$ to the reference element $K := \left\{(x_1,x_2) \in \mathbb{R}^2; 0 \le x_1 \le 1\,, 0 \le x_2 \le 1 - x_1\right\}$ defined by



$$x_1 := \xi,$$
$$x_2 := \eta(1-\xi).$$

This degenerate mapping can be used to transform the integration of a function $f : K \mapsto \mathbb{R}$ over the reference element K to an integration over the unit square Q. This approach is known as Duffy trick and can be used to generate quadrature rules of arbitrary order on the reference triangle.

(9.2a) Compute the Jacobian of the mapping $\phi : Q \mapsto K$ and show that the following transformation formula holds true

$$\int_Q f(\xi, \eta(1-\xi))(1-\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta = \int_K f(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 \tag{9.2.1}$$

for an arbitrary function $f : K \mapsto \mathbb{R}$.

(9.2b) On the unit square one can use *tensor product Gauss-Legendre* quadrature formulas, which are obtained as follows, see also [NPDE, Ex. 3.6.95]: If $\zeta_\ell \in]0,1[$, $\omega_\ell \in \mathbb{R}$, $\ell = 1, \ldots, n$, are the nodes and weights, respectively, of the n-point Gauss-Legendre quadrature rule on $[0,1]$, then we approximate

$$\int_Q g(x_1, x_2)\,\mathrm{d}\boldsymbol{x} \approx \sum_{j=1}^{n}\sum_{\ell=1}^{n} \omega_j \omega_\ell g(\zeta_j, \zeta_\ell)\,. \tag{9.2.2}$$

Show that this quadrature rule is of (maximal) order $2n$.

HINT: The order of a quadrature rule has been introduced in [NPDE, Def. 3.6.87]. It is known that the n-point Gauss-Legendre quadrature rule on an interval is of maximal order $2n$.

(9.2c) We consider the quadrature formula on the unit triangle generated by the Duffy trick from the n^2-point tensor product Gauss-Legendre quadrature formula on the unit square presented in subproblem (9.2b). Determine the (maximal) order of this rule.

HINT: According to [NPDE, Def. 3.6.87] you have to check, which integrals $\int_K x_1^p\, x_2^q\,\mathrm{d}x$ of monomials with $p + q \leq m$ are integrated exactly. [NPDE, Lemma 3.6.61] is useful for that purpose.

Problem 9.3 Parametric Finite Elements on Curved Triangles (Core problem)

In [NPDE, Section 3.7.4] we have seen how the paradigm of parametric finite elements can be used to deal with *curved elements*, which is important for higher order resolution of boundaries and interfaces. In this problem we study this technique for "triangular" elements whose edges are pieces of parabolas.

Taking the cue of [NPDE, § 3.7.30], we consider the quadratic transformation

$$\begin{aligned}\boldsymbol{\Phi}_{\tilde{K}}(\hat{\boldsymbol{x}}) :=& \boldsymbol{a}^1\hat{\lambda}_1(\hat{\boldsymbol{x}}) + \boldsymbol{a}^2\hat{\lambda}_2(\hat{\boldsymbol{x}}) + \boldsymbol{a}^3\hat{\lambda}_3(\hat{\boldsymbol{x}}) + \\ & \boldsymbol{d}^3 4\hat{\lambda}_1(\hat{\boldsymbol{x}})\hat{\lambda}_2(\hat{\boldsymbol{x}}) + \boldsymbol{d}^1 4\hat{\lambda}_2(\hat{\boldsymbol{x}})\hat{\lambda}_3(\hat{\boldsymbol{x}}) + \boldsymbol{d}^2 4\hat{\lambda}_1(\hat{\boldsymbol{x}})\hat{\lambda}_3(\hat{\boldsymbol{x}})\end{aligned} \tag{9.3.1}$$

mapping the reference triangle (unit triangle) $\widehat{K}$ to the triangle $\tilde{K}$ with curved edges, as shown in Figure 9.1. Here $\hat{\lambda}_i$, $i = 1, 2, 3$, are the barycentric coordinate functions on $\widehat{K}$, and $\boldsymbol{a}^i$, $\boldsymbol{d}^i$ are given vectors $\in \mathbb{R}^2$.

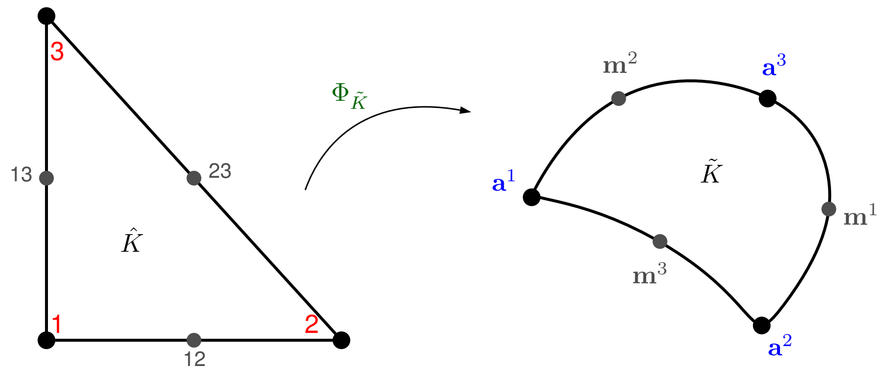


Figure 9.1

(9.3a) Determine the location of the vertices of $\tilde{K}$ in terms of $\boldsymbol{a}^i$, $\boldsymbol{d}^i$.

(9.3b) Give a geometric interpretation of the vectors $\boldsymbol{d}^i$.

HINT: Rely on the image points of the midpoints of the edges of $\widehat{K}$ and use the notations from Figure 9.1.

(9.3c) The mapping $\widetilde{\boldsymbol{\Phi}}_K$ from [NPDE, Eq. (3.7.31)] in [NPDE, § 3.7.30] is a special version of (9.3.1). Determine the corresponding values for $\boldsymbol{a}^i$ and $\boldsymbol{d}^i$ using the local numbering as in [NPDE, Fig. 153].

(9.3d) Compute the Jacobian matrix $D\boldsymbol{\Phi}_{\tilde{K}}$.

(9.3e) Compute the determinant of Jacobian matrix $|\det D\boldsymbol{\Phi}_{\tilde{K}}|$.

(9.3f) Write a C++ Method

```
template <class Function>
void assemLocQuadTri(std::vector<Coordinate> const & a,
                     std::vector<Coordinate> const & d,
                     Function const& alpha, ElementMatrix & local);
```

which computes the element matrix for

$$\int_{\Omega} \boldsymbol{\alpha}(\mathbf{x})\, \mathbf{grad}\, \mathbf{u}(\mathbf{x}) \cdot \mathbf{grad}\, \mathbf{v}(\mathbf{x})\, \mathrm{d}\mathbf{x},$$

where $\boldsymbol{\alpha}(\mathbf{x})$ is a scalar coefficient. Use *parametric* piecewise linear Lagrangian finite elements and the (transformed) edge midpoint quadrature rule from Problem 9.1.

The method takes as inputs an arbitrary function alpha(Coordinate const & x), and the vectors a and d (containing $\boldsymbol{a}^i$ and $\boldsymbol{d}^i$ respectively).

Recall Coordinate corresponds to Dune::FieldVector<calc_t,world_dim> and ElementMatrix corresponds to AnalyticalLocalMass::ElementMatrix in your implementation of subproblem (7.4b). You can assume that ElementMatrix local has been initialized and set to zero before it is passed to the method.

HINT: You may in addition use Eigen::Matrix2d for the Jacobian, as it is equipped with the methods inverse() and determinant().

HINT: A template file is available in the lecture `svn` repository

```
assignments_codes/assignment9/Problem3
```

There you will also find a testcall written in `main.cc` and its output in `testcall_output.txt`

Problem 9.4 Poisson Equation on a Disk

This problem offers a comprehensive treatment of analytical and algorithmic techniques for parametric linear finite elements on general hybrid meshes (with straight edges), see, in particular [NPDE, Section 3.7.2]. It also revisits assembly of Galerkin matrices from element matrices.

We consider the homogeneous Dirichlet problem for the Laplacian Δ:

$$-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{9.4.1}$$

where Ω is the unit disk

$$\Omega := \left\{ \boldsymbol{x} \in \mathbb{R}^2 \,\middle|\, \|\boldsymbol{x}\| < 1 \right\}.$$

Again, a student does not want to use any finite element library, but prefers to write a code for this boundary value problem from scratch. Warned by his buddy that polar coordinates and a finite difference discretization caused terrible difficulties, he opts for a finite element method on a *hybrid mesh* comprising quadrilaterals and triangles like that depicted in Figure 9.2: the nodes of the mesh are the origin and the intersection points of circles with radii $\frac{j}{N}$, $j = 1, \dots, N$, with rays at angles $\frac{2\pi j}{N}$, $j = 1, \dots, N$. He settles for a polygonal approximation of $\partial\Omega$ as in Figure 9.2.

On the hybrid mesh the student wants to employ a finite element Galerkin discretization of (9.4.1) based on continuous trial/test function that are

- piecewise linear on the triangles ($\rightarrow$ [NPDE, Section 3.3])
- parametrically mapped bilinear functions on the quadrilaterals ($\rightarrow$ [NPDE, Section 3.7.2]).

As explained in class, for the quadrilaterals the mapping from the unit square is a bilinear transformation, see [NPDE, Eq. (3.7.16)].

(9.4a) Give the coefficients of the bilinear transformation

$$\boldsymbol{\Phi}_K(\widehat{\boldsymbol{x}}) = \begin{pmatrix} \alpha_1 + \beta_1 \widehat{x}_1 + \gamma_1 \widehat{x}_2 + \delta_1 \widehat{x}_1 \widehat{x}_2 \\ \alpha_2 + \beta_2 \widehat{x}_1 + \gamma_2 \widehat{x}_2 + \delta_2 \widehat{x}_1 \widehat{x}_2 \end{pmatrix}, \quad \alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{R},$$

for the quadrilateral formed by the circles with radii $\frac{j}{N}$ and $\frac{j+1}{N}$, $1 \le j < N$, and the rays with angles $\frac{2\pi m}{N}$ and $\frac{2\pi(m+1)}{N}$, $0 \le m < N$, see Figure 9.3.

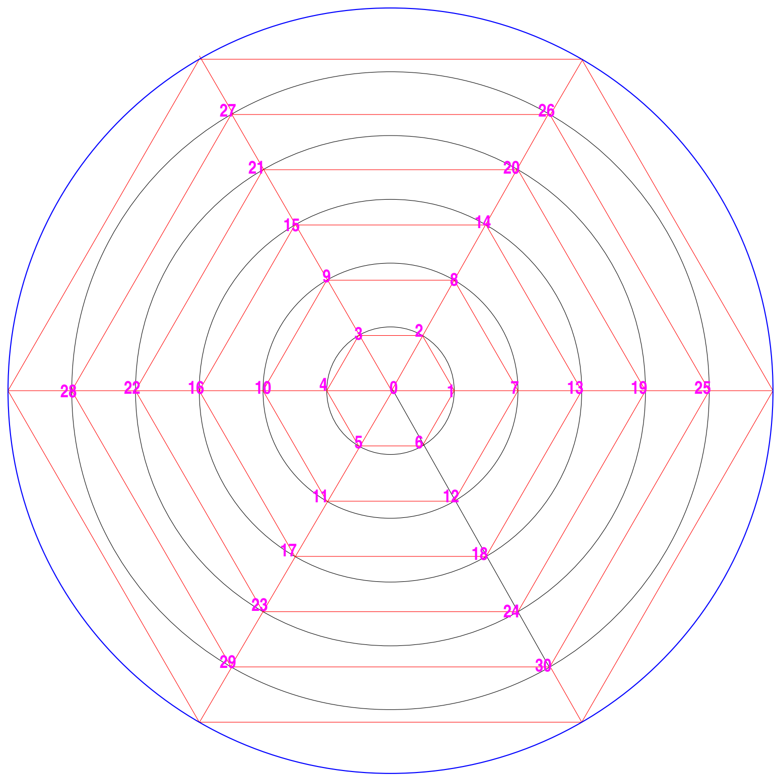


Figure 9.2: Hybrid mesh with piecewise linear approximation of $\partial\Omega$. The red lines are the edges of the mesh.

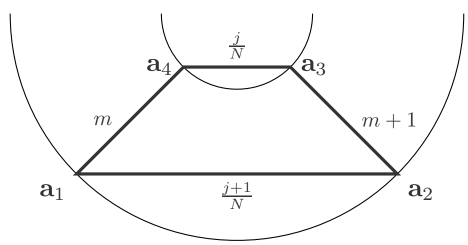


Figure 9.3: quadrilateral element

(9.4b) Compute the Jacobian of the bilinear transformation from the subproblem (9.4a) and its determinant. Both are functions on the unit square, see [NPDE, Ex. 3.7.27].

(9.4c) For the computation of the element (load) vectors the student chooses a simple one-point quadrature formula based on the center of gravity (of a triangle or the unit square, respectively). Based on this choice compute the element (load) vector for the general quadrilateral from subproblem (9.4a).

(9.4d) The same one-point quadrature as in subproblem (9.4c) is used for the evaluation of the element (stiffness) matrices. Compute the element matrix for the quadrilateral considered in subproblem (9.4a).

(9.4e) What is the dimension of the finite element space $V_{0,N} \subset H_0^1(\Omega)$?

(9.4f) Given the mesh in Figure 9.2 with the finite element space already described, sketch the structure of the Galerkin matrix $\mathbf{A}$. For this, number the nodes counterclockwise from the inner part of the mesh to the exterior as indicated in the figure. Do you observe any pattern?

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References

[NPDE] Lecture Slides for the course "Numerical Methods for Partial Differential Equations".SVN revision # 75265.

[NCSE] Lecture Slides for the course "Numerical Methods for CSE".

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