Course 401-0674-00L: NPDE

Endterm, 31-05-2013

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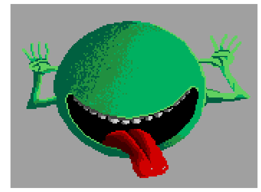

Don't panic!
Good luck!

Name

Student number

Points

Problem 1 Discretization error for linear and quadratic Lagrangian finite elements [5 points]

On a polygonal, bounded domain $\Omega \subset \mathbb{R}^2$ we consider the finite element Galerkin discretization of the boundary value problem

$$-\Delta u + u = f \in L^2(\Omega) \quad \text{in } \Omega \subset \mathbb{R}^2, \quad u = 0 \quad \text{on } \partial\Omega\,. \tag{1.1}$$

by means of piecewise linear Lagrangian finite elements (FE space $\mathcal{S}^0_{1,0}(\mathcal{M})$) and piecewise quadratic Lagrangian finite elements (FE space $\mathcal{S}^0_{2,0}(\mathcal{M})$) on a triangular mesh $\mathcal{M}$. The respective finite element solutions will be denoted by $u_L \in \mathcal{S}^0_{1,0}(\mathcal{M})$ and $u_Q \in \mathcal{S}^0_{2,0}(\mathcal{M})$.

(1a) [3 points] Show that

$$\|u - u_Q\|^2_{\mathsf{a}} + \|u_Q - u_L\|^2_{\mathsf{a}} = \|u - u_L\|^2_{\mathsf{a}}\,, \tag{1.2}$$

$u \in H^1_0(\Omega)$ is the exact solution and $\|\cdot\|_{\mathsf{a}}$ stands for the energy norm induced by the variational formulation of (1.1).

(1b) [2 points] Give an argument, why

$$\|u - u_Q\|_{\mathsf{a}} \leq \|u - u_L\|_{\mathsf{a}} \tag{1.3}$$

holds true.

Problem 2 Convergence of finite element solutions [6 points]

On the "L-shaped" domain $\Omega =]-1,1[^2 \backslash [-1,0]^2$ we consider the second-order elliptic boundary value problem

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega \; . \tag{2.1}$$

In a code a Galerkin discretization by means of piecewise linear and quadratic Lagrangian finite elements is employed.

(2a) [3 points] Consider the case when f and g are set to produce the exact solution $u(\boldsymbol{x}) = \cos(\pi x_1)\cos(\pi x_2)$.

Describe in qualitative and quantitative terms the convergence of the finite element solutions in the energy norm on a sequence of triangular meshes created by successive regular refinement of some initial mesh.

(2b) [3 points] Somebody else uses the code on the boundary value problem (2.1) for $f \equiv 1$ and $g = 0$ and he observes the errors in energy norm displayed in Figure 2.1 for the finite element solutions on a sequence of triangular meshes created by successive regular refinement of some initial mesh.

Explain, why the answer to sub-problem (2a) completely fails to match the observations in this case.

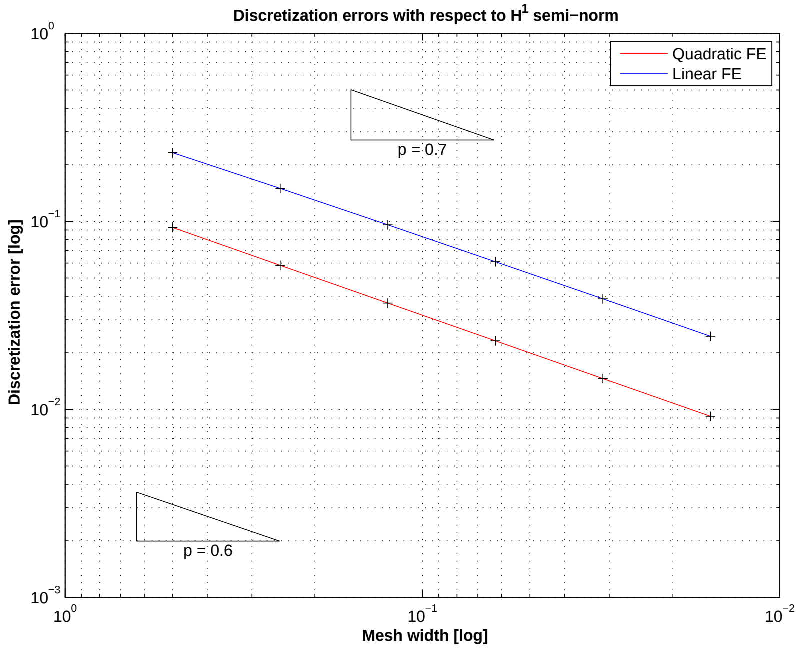

Figure 2.1: Energy norm of discretization errors for both linear and quadratic Lagrangian finite elements.

Problem 3 Linear output functionals [6 points]

Which of the following output functionals are linear and well defined on $L^2(\Omega)$ and $H^1(\Omega)$, respectively, for $\Omega = \{\boldsymbol{x} \in \mathbb{R}^2 : \|\boldsymbol{x}\| < 1\}$? Answer by entering "YES" or "NO" in the blank fields of the table.

functional	linear?	defined on $L^2(\Omega)$?	defined on $H^1(\Omega)$?
$J(v) = \int_\Omega \mathbf{c} \cdot \mathbf{grad}\, v(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}$, $\mathbf{c} \in \mathbb{R}^2$			
$J(v) := \int_{\partial\Omega} \mathbf{grad}\, v(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x}) \,\mathrm{d}S(\boldsymbol{x})$			
$J(v) := \lvert v(\boldsymbol{x}_0) \rvert$, $\boldsymbol{x}_0 \in \Omega$			
$J(v) := \int_\Omega \mathbf{c} v(\frac{\boldsymbol{x}}{\lVert \boldsymbol{x} \rVert}) \,\mathrm{d}\boldsymbol{x}$, $\mathbf{c} \in \mathbb{R}^2$			

Problem 4 Parabolic evolution [5 points]

For testing purposes one considers the parabolic evolution problem

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u &= 0 \quad \text{in } \Omega \times]0, T[\ , \\ u &= 0 \quad \text{on } \partial\Omega \times]0, T[\ , \\ u(\boldsymbol{x}, 0) &= u_0(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in \Omega\ , \end{aligned} \tag{4.1}$$

on the unit square $\Omega =]0, 1[^2$. Choosing $u_0(\boldsymbol{x}) = \sin(\pi x_1) \sin(\pi x_2)$ one obtains $u(\boldsymbol{x}, t) = \exp(-\pi^2 t) u_0(\boldsymbol{x})$ as exact solution.

A method of lines approach is employed: Discretization in space relies on quadratic Lagrangian finite elements, whereas discretization in time is done using an L-stable SDIRK implicit Runge-Kutta scheme of order 2 with uniform timestep $\tau > 0$.

(4a) [3 points] For fixed timestep τ we examine the $L^2(\Omega)$-norm of the discretization error at final time $T = \frac{1}{2}$ for an (infinite) sequence of meshes created by uniform regular refinement. Indicate the qualitative dependence of this error norm on the mesh-width h by drawing a suitable error curve in Figure 4.1.

HINT: Assume an error norm of 1 on the coarsest mesh.

(4b) [2 points] Now we track the error norm $E(t_j) := \lVert u(t_j) - u_N(t_j) \rVert_{L^2(\Omega)}$ as a function of $t_j = j\tau$, $j \in \mathbb{N}$, for fixed finite element mesh and fixed timestep τ. What can we expect? Sketch E in Figure 4.2, assuming $E(0) = 0.2$.

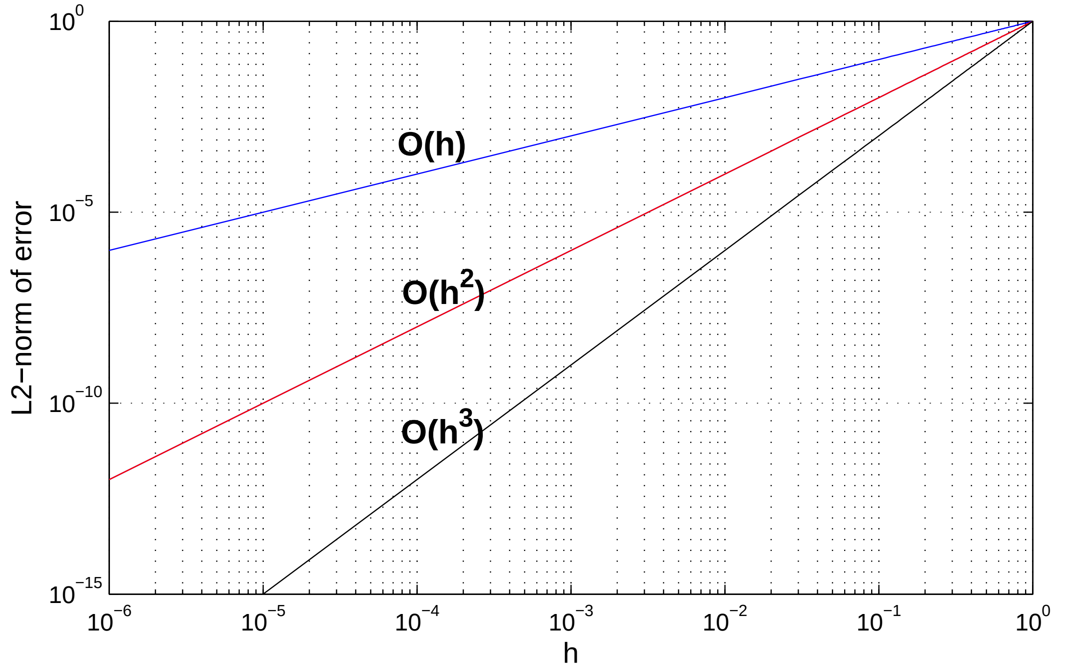

Figure 4.1: Empty double logarithmic coordinate system, mesh-width h versus $\|u(T) - u_N(T)\|_{L^2(\Omega)}, T > 0$ fixed.

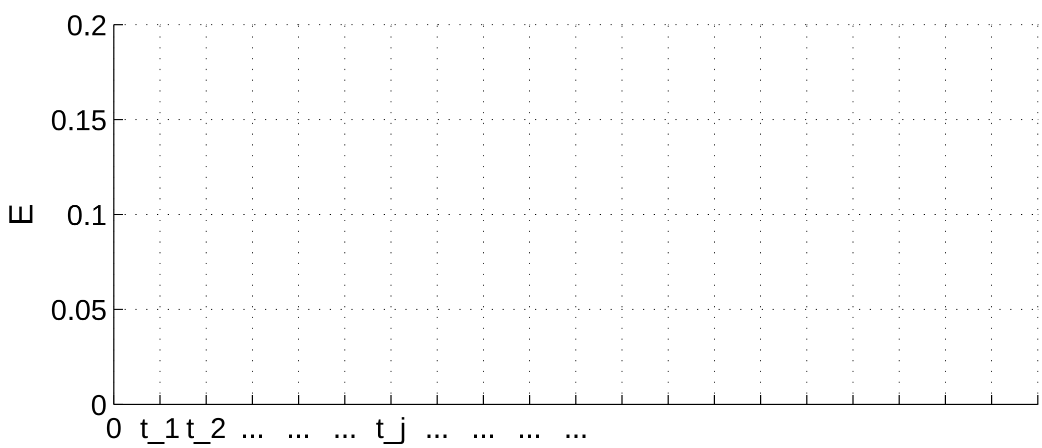

Figure 4.2: Empty linear coordinate system discrete times $t_1, t_2, ...t_j, ...$ vs. E. Timestep τ and mesh fixed.

Problem 5 Singular perturbations [3 points]

Explain the concept of singular perturbation of a boundary value problem for the BVP

$$-\epsilon\Delta u + \mathbf{v} \cdot \mathbf{grad}\, u = 0 \quad \text{in } \Omega, \quad u = g \quad \text{on } \partial\Omega\,, \tag{5.1}$$

as $\epsilon \to 0$. Here Ω is a domain in $\mathbb{R}^2$ and $\mathbf{v} \in \mathbb{R}^2 \setminus \{0\}$.