Course 401-0674-00L: Numerical Methods for Partial Differential Equations

Examination, 06.08.2012

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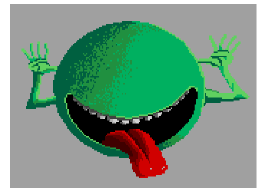


Dont't panic !
Good luck !

Duration of examination: 180 minutes

The total number of points is 350. A full grade can be achieved with significantly fewer points. Please pay attention to the number of points awarded for each (sub-)task. It is roughly correlated with the amount of information your answer should contain. For additional information see the examination instruction sheet.

Problem 1. Parabolic Evolution Problem [90 points]

Let $\Omega \subset \mathbb{R}^2$ and the time-dependent function $u : [0, T] \to H^1(\Omega)$ solve the following variational formulation of an evolution problem posed on the space-time cylinder $\Omega \times [0, T]$:

$$\int_\Omega \mathbf{grad}\, u(\boldsymbol{x}, t) \cdot \mathbf{grad}\, v(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} + \frac{\mathrm{d}}{\mathrm{d}t} \int_{\partial\Omega} u(\boldsymbol{x}, t) v(\boldsymbol{x})\, \mathrm{d}S(\boldsymbol{x}) = 0 \quad \forall v \in H^1(\Omega) \,, \qquad u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \Omega \,. \tag{1}$$

Note that the bilinear form with the time derivative in front is an integral over the boundary $\partial\Omega$.

(1a) (10 points) Show that $t \mapsto |u(\cdot, t)|_{H^1(\Omega)}$ is non-increasing.

Hint: You can take for granted that $v = \frac{\partial}{\partial t} u$ is a valid test function for the variational equation in (1).

(1b) (5 points) The spatial Galerkin semi-disretization of (1) results in an ordinary differential equation (ODE) of the form

$$\boldsymbol{A}\vec{\boldsymbol{\mu}}(t) + \boldsymbol{B}\frac{d}{dt}\vec{\boldsymbol{\mu}}(t) = 0, \qquad \vec{\boldsymbol{\mu}}(t) = \vec{\boldsymbol{\mu}}_0 \,. \tag{2}$$

What are the formulas for the entries of the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, if the basis $\{b_N^j\}_{j=1}^N$, of the N-dimensional trial and test space $V_N \subset H^1(\Omega)$ is used?

(1c) (5 points) (Depends on (1b))

Which properties of the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ introduced in sub-problem (1b) are ensured regardless of the choice of the trial and test space V_N and of the basis $\{b_N^j\}_{j=1}^N$?

(1d) (5 points) (Depends on (1b))

Now we focus on the specific choice $V_N = \mathcal{S}_1^0(\mathcal{M})$, $\mathcal{M}$ a triangular mesh of Ω, for the Galerkin finite element semi-discretization of (1).

Which difficulty is encountered when trying to solve the resulting ODE of the form (2) by means of the explicit Euler timestepping scheme?

(1e) (10 points) SDIRK-2 timestepping is an L-stable implicit 2-stage Runge-Kutta method described by the Butcher scheme

$$\begin{array}{c|cc} \lambda & \lambda & 0 \\ 1 & 1-\lambda & \lambda \\ \hline & 1-\lambda & \lambda \end{array} \quad , \quad \lambda := 1 - \tfrac{1}{2}\sqrt{2} > 0. \tag{3}$$

Which equations have to be solved in every timestep, when SDIRK-2 with timestep $\tau > 0$ is applied to the ODE (2)?

(1f) (10 points) (Depends on (1b))

Why is SDIRK-2 for (2) feasible for any timestep $\tau > 0$, any trial and test space $V_N \neq \{0\}$ and any choice of its basis?

HINT: You have to show that the linear systems of equations to be solved to obtain the Runge-Kutta increments always have a solution.

(1g) (10 points) The LehrFEM MATLAB function

```
function u = solveRobinBVP(mesh,g)
```

implements a the finite element Galerkin discretization of the boundary value problem

$$-\Delta u = 0 \quad \text{in } \Omega \,, \quad u + \mathbf{grad}\, u \cdot \boldsymbol{n} = g \quad \text{on } \partial\Omega \,, \tag{4}$$

using the finite element space $V_N = \mathcal{S}_1^0(\mathcal{M})$ on a triangular mesh $\mathcal{M}$, passed as `mesh` argument to the function. The argument `g` is a handle of the type `@(x)` providing the continuous function $g : \mathbb{R}^2 \to \mathbb{R}$. The function returns the coefficient vector of the Galerkin solution with respect to the standard nodal basis of $\mathcal{S}_1^0(\mathcal{M})$.

Reusing parts of `solveRobinBVP` develop a LehrFEM MATLAB function

```
function [A,B] = compGalMats(mesh)
```

that computes the Galerkin matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ introduced in sub-problem (1b) for the trial and test space $V_N = \mathcal{S}_1^0(\mathcal{M})$ equipped with the standard nodal basis. Here $\mathcal{M}$ is a triangular mesh of Ω passed as argument `mesh` (a LehrFEM mesh data type complete with edge information).

(1h) (15 points) (Depends on (1e))

Implement a MATLAB function

```
mufinal = RadTEvl(u0,mesh,Tfinal,m)
```

that carries out m uniform timesteps of the L-stable SDIRK-2 implicit 2-stage Runge-Kutta method from (3) of sub-problem (1e). Spatial discretization should rely on $V_N = \mathcal{S}_1^0(\mathcal{M})$, $\mathcal{M}$ a triangular mesh of Ω passed as argument `mesh` (a LehrFEM mesh data type complete with edge information). The column vector `u0` supplies the initial value $\vec{\boldsymbol{\mu}}_0 \in \mathbb{R}^N$, $N := \dim V_N$, `Tfinal` the end time T. The function should return an approximation of $u(\boldsymbol{x}, t)$ for $t = T$ in the form of a coefficient (column) N-vector `mufinal` with respect to the nodal basis of the finite element space.

HINT: A scrambled MATLAB implementation of `compGalMats` from sub-problem (1g) is available in the file `compGalMats_ref.p`.

(1i) (10 points) (Depends on (1h))

Copy your implementation of `RadTEvl` from sub-problem (1h) to a file `RadTEvlNorm.m` and extend it to a function

```
[mufinal,H1seminorms] = RadTEvlNorm(u0,mesh,Tfinal,m),
```

which is supposed to return approximations of $|u(\cdot, t_l)|_{H^1(\Omega)}$, $t_l := l\frac{T}{m}$, $l = 0, \dots, m$, in the column vector `H1seminorms`, in addition to an approximate solution at final time. The arguments of the function are explained in sub-problem (1h).

(1j) (10 points) Write a MATLAB script

```
plotenergyevolution
```

that plots the approximate values for $|u(\cdot, t_l)|_{H^1(\Omega)}$, $t_l := l\frac{T}{m}$, $l = 0, \dots, m$ versus time.

Use the initial function $u(\boldsymbol{x}) = \sin(30x_1) + \sin(30x_2)$, $T = 6$, $m = 200$ and `mesh` read from `Coord_Circ.dat` and `Elem_Circ.dat`.

Choose a suitable plot that reveals a potential exponential decay of $t \mapsto |u(\cdot, t_l)|_{H^1(\Omega)}$.

HINT: A scrambled reference implementation of `RadTEvlNorm` is supplied in the file `RadTEvlNorm_ref.p`.

Problem 2. Discontinuous Galerkin for 1D Conservation Laws [130 points]

We consider the Cauchy problem for the scalar non-liner conservation law

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0 \quad &\text{in } \mathbb{R} \times]0, T[\\ u(x, 0) = u_0(x) \quad &\text{for } x \in \mathbb{R} \;, \end{aligned} \tag{5}$$

with smooth flux function $f : \mathbb{R} \mapsto \mathbb{R}$ and initial data u_0 compactly supported in $[0, 1]$.

Based on the infinite equidistant spatial mesh $\mathcal{M} := \{]x_{j-1}, x_j[: \ x_j = hj, \ j \in \mathbb{Z}\}$ with meshwidth $h > 0$ we define the function space

$$V_N = \{v \in L^2(\Omega) : v\big|_{[x_{j-1}, x_j]} \in \mathcal{P}_1\} \tag{6}$$

of *discontinuous* piecewise linear functions on $\mathcal{M}$ (see figure 1 for an example)

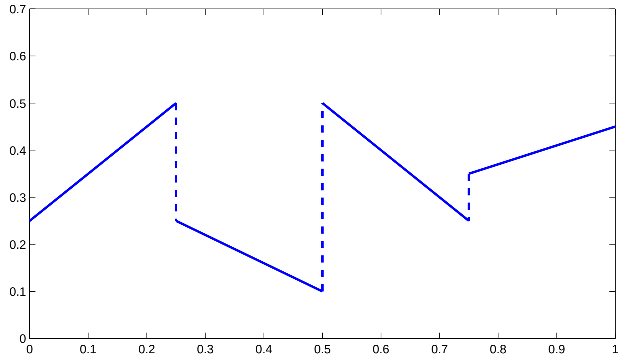


Figure 1: A sample discontinuous piecewise linear function.

(2a) (10 points) Show that a continuous solution u of (5) satisfies

$$\int_{x_{j-1}}^{x_j} \frac{\partial u}{\partial t}(x,t)\, v_N(x)\,\mathrm{d}x - \int_{x_{j-1}}^{x_j} f(u(x,t))\frac{\mathrm{d}v_N}{\mathrm{d}x}(x)\,\mathrm{d}x + \\ f(u(x_j,t))v_N(x_j-) - f(u(x_{j-1},t))v_N(x_{j-1}+) = 0 \quad \forall v_N \in V_N\ , j \in \mathbb{Z}\,. \tag{7}$$

Here, we adopted the notation $v(x_j\pm) := \lim\limits_{\delta\to 0^+} v(x_j \pm \delta)$ in order to resolve the ambiguity of $v(x_j)$.

HINT: Perform the usual steps in the derivation of a spatial variational formulation: test, integrate, and integrate by parts (in space).

Next, we aim for a spatial Galerkin semi-discretization of (7) based on the space V_N. However, plugging $u_N \in V_N$ into (7) instead of u faces the problem of ambiguity of $f(u(x_j,t))$. Therefore $f(u(x_j,t))$ in (7) is replaced with a *numerical flux* $F(u(x_j-), u(x_j+)))$, which leads to the semi-discrete *discontinuous Galerkin variational problem*: seek $u_N : [0,T] \mapsto V_N$ such that

$$\int_{-\infty}^{\infty} \frac{\partial u_N}{\partial t}(x,t)\, v_N(x)\,\mathrm{d}x - \sum_{j=\infty}^{\infty}\int_{x_{j-1}}^{x_j} f(u_N(x,t))\frac{\mathrm{d}v_N}{\mathrm{d}x}(x)\,\mathrm{d}x + \\ \sum_{j=-\infty}^{\infty} F(u_N(x_j-,t), u_N(x_j+,t))(v_N(x_j-) - v_N(x_j+)) = 0 \quad \forall v_N \in V_N\,. \tag{8}$$

(2b) (10 points) We know *a priori* that u_0 is compactly supported in $[0,1]$, $0 \le u_0(x) \le 1$ and that $|f'(u)| \le c$ for all $0 \le u \le 1$. If $T = 1$, which is the smallest truncated spatial mesh

$$\hat{\mathcal{M}} := \{]x_{j-1}, x_j[:\ x_j = hj,\ j \in \{-M_l, \dots, M_r\}\} \tag{9}$$

that allows the spatial discretization of (5) without any impact of the truncation? Find the numbers $M_l, M_r \in \mathbb{N}_0$.

(2c) (15 points) We denote by $\hat{V}_N$ the space V_N restricted to the spatial interval covered by the truncated spatial mesh $\hat{\mathcal{M}}$ from (9). As basis of $\hat{V}_N$ we choose

$$\left\{ b_1^{-M_l}, b_2^{-M_l}, b_1^{-M_l+1}, b_2^{-M_l+1}, \dots, b_1^{M_r-1}, b_2^{M_r-1}, b_2^{M_r}, b_2^{M_r} \right\}\ , \tag{10}$$

where

$$b_1^j(x) = \begin{cases} 1 & \text{for } x_{j-1} \leq x < x_j \;, \\ 0 & \text{elsewhere,} \end{cases} \quad , \quad b_2^j(x) = \begin{cases} x - \frac{1}{2}(x_{j-1}+x_j) & \text{for } x_{j-1} \leq x < x_j \;, \\ 0 & \text{elsewhere.} \end{cases} \tag{11}$$

In (8) we expand u_N into these basis functions on the truncated mesh $\hat{\mathcal{M}}$ with meshwidth h, testing with $v_N = b_1^j$ and $v_N = b_2^j$. This leads to an ordinary differential equation (ODE) for the time-dependent coefficient vector $\vec{\boldsymbol{\mu}} = \vec{\boldsymbol{\mu}}(t) \in \mathbb{R}^N$, $N := 2(M_l + M_r + 1)$, of the semi-discrete Galerkin solution, assuming the ordering of basis functions given in (10). This ODE takes the form

$$\boldsymbol{B}\frac{\mathrm{d}}{\mathrm{d}t}\vec{\boldsymbol{\mu}} + G(\vec{\boldsymbol{\mu}}) = 0 \;, \tag{12}$$

with a matrix $\boldsymbol{B} \in \mathbb{R}^{N,N}$ and potentially non-linear function $G : \mathbb{R}^N \mapsto \mathbb{R}^N$.

Write an *efficient* MATLAB function

```
function B = compBmat(h,Ml,Mr)
```

that computes the matrix $\boldsymbol{B}$ given M_l, M_r, and h.

(2d) (20 points) Give a formula for the function G from (12) in terms of a general flux function f and two-point numerical flux $F = F(v,w)$. Simple point evaluations and integrals of f and F may be used.

HINT: If it helps, you can use the notation $\mu_{i,1}$ and $\mu_{i,2}$ to reference the elements in a vector $\vec{\boldsymbol{\mu}}$ corresponding to basis functions b_1^i and b_2^i respectively (e.g. as if $\boldsymbol{\mu}$ were a matrix).

(2e) (10 points) (Depends on (2d))

Now use the two point Gaussian quadrature rule

$$\int_0^1 \varphi(\xi)\,\mathrm{d}\xi \approx \frac{1}{2}\left(\varphi(\tfrac{1}{2}(1-\tfrac{1}{\sqrt{3}})) + \varphi(\tfrac{1}{2}(1+\tfrac{1}{\sqrt{3}}))\right) \tag{13}$$

to approximate all integrals occurring in the expression for G obtained in the previous sub-problem. State the resulting formula.

HINT: Do not forget the rescaling of quadrature weights when using (13) on an interval of length h.

(2f) (20 points) (Depends on (2d) and (2e))

Write a MATLAB function

```
function gvec = G(muvec,f,F,Ml,Mr,h)
```

that evaluates $G(\vec{\boldsymbol{\mu}})$ for a coefficient vector $\vec{\boldsymbol{\mu}}$ passed in `muvec`, on a finite equidistant mesh $\hat{\mathcal{M}}$ described by M_l, M_r, and h. The arguments `f` and `F` contain function handles to the flux function $f = f(u)$ and the numerical flux $F = F(v,w)$, respetively. 2-point Gaussian quadrature as in the previous sub-problem is to be used.

HINT: Use the MATLAB function `reshape` to work with $2 \times N/2$-matrices instead of $N \times 1$-vectors to make your code easier to read (and write), cf. the hint for (2d).

(2g) (15 points) We rely on the spatial discontinuous Galerkin semi-discretization on a truncated mesh $\hat{\mathcal{M}}$ described above. Write a MATLAB function

```
function mufinal = dgcl(mu0,f,F,T,Ml,Mr,h,m)
```

that uses m uniform timesteps of the explicit 2-stage Runge-Kutta timestepping scheme described by the Butcher scheme

$$\begin{array}{c|cc} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \hline & 0 & 1 \end{array} , \tag{14}$$

to compute an approximation of $u(x,T)$ for final time $T > 0$. `mu0` passes the coefficient vector of an approximation of u_0. All other arguments play the same roles as in sub-problems (2e) and (2f).

HINT: Scrambled reference implementations of `compBmat` and `G` are available in the files `compBmat_ref` and `G_ref`.

(2h) (15 points) From now on we consider the traffic flow problem with $f(u) = u(1-u)$ and use the so-called Engquist-Osher numerical flux

$$F_{\mathrm{EO}}(v,w) := \tfrac{1}{2}(f(v) + f(w)) - \tfrac{1}{2}\int_v^w |f'(\xi)| \, \mathrm{d}\xi \; . \tag{15}$$

Implement this numerical flux for $f(u) = u(1-u)$ as a MATLAB function

```
function F = Feo(v,w) .
```

(2i) (15 points) Use the fully discrete discontinuous Galerkin solver implemented in `dgcl` from sub-problem (2h) with Engquist-Osher numerical flux and $f(u) = u(1-u)$ to solve the evolution problem over the time interval $[0,1]$ with initial data

$$u_0(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq 1 \; , \\ 0 & \text{elsewhere} \; . \end{cases} \tag{16}$$

As spatial mesh width use $h = 0.05$, timestep $\tau = h/3$, and truncate the mesh to $[-2,2]$.

Plot the spatial cell averages of the obtained approximation of $u(x,1)$ versus the spatial variable x.

All this should be accomplished by the MATLAB script

```
solveTrafficFlow.
```

HINT: Reference implementations of `dgcl` and `Feo` are available in the file `dgcl_ref.p` and `Feo_ref.p`. Take care to get `mu0` right.

Problem 3. Basis transformation [50 points]

Let a polygon $\Omega \subset \mathbb{R}^2$ be equipped with a triangular mesh $\mathcal{M}$, whose vertices (set $\mathcal{V}(\mathcal{M})$) and edges (set $\mathcal{E}(\mathcal{M})$) are numbered from 1 to $\sharp\mathcal{V}(\mathcal{M})$ and $\sharp\mathcal{E}(\mathcal{M})$, respectively.

We consider the space $V_N = \mathcal{S}_2^0(\mathcal{M})$ of quadratic Lagrangian finite element functions on $\mathcal{M}$. A basis of this space is given by the standard nodal basis.

$$\mathcal{B}_q = \{\widetilde{b}_N^j\}_{j=1}^N, \qquad\qquad N = \sharp\mathcal{V}(\mathcal{M}) + \sharp\mathcal{E}(\mathcal{M})\ ,$$

where we number the $\sharp\mathcal{V}(\mathcal{M})$ vertex associated basis functions before the $\sharp\mathcal{E}(\mathcal{M})$ edge associated basis functions.

Another basis for V_N is the hierarchical basis given by

$$\mathcal{B}_h = \{\widehat{b}_N^j\}_{j=1}^N, \qquad\qquad \widehat{b}_N^j = \begin{cases} b_N^j, & j = 1, \dots, \sharp\mathcal{V}(\mathcal{M})\ , \\ \widetilde{b}_N^j, & j = \sharp\mathcal{V}(\mathcal{M}), \dots, \sharp\mathcal{V}(\mathcal{M}) + \sharp\mathcal{E}(\mathcal{M})\ , \end{cases}$$

where $\{b_N^j\}_{j=1}^{\sharp v}$ is the nodal basis for the space $\mathcal{S}_1^0(\mathcal{M})$ of linear Lagrangian finite element functions, and $\widetilde{b}_N^j$ are the edge associated nodal basis functions from $\mathcal{B}_q$.

(3a) (20 points) Let $\vec{\boldsymbol{\mu}}_q$ be the coefficient vector of a function $u_N \in V_N$ w.r.t. the nodal basis $\mathcal{B}_q$. Further let $\vec{\boldsymbol{\mu}}_h$ be the coefficient vector of the *same* function u_N w.r.t. the hierarchical basis $\mathcal{B}_h$.

For the mesh $\mathcal{M}$ depicted in figure 2 find the matrix $\boldsymbol{S} \in \mathbb{R}^{N,N}$, $N = 4 + 5$ such that $\boldsymbol{\mu}_h = \boldsymbol{S}\boldsymbol{\mu}_q$. Use the numbering of the basis functions as indicated by edge and vertex numbers in figure 2.

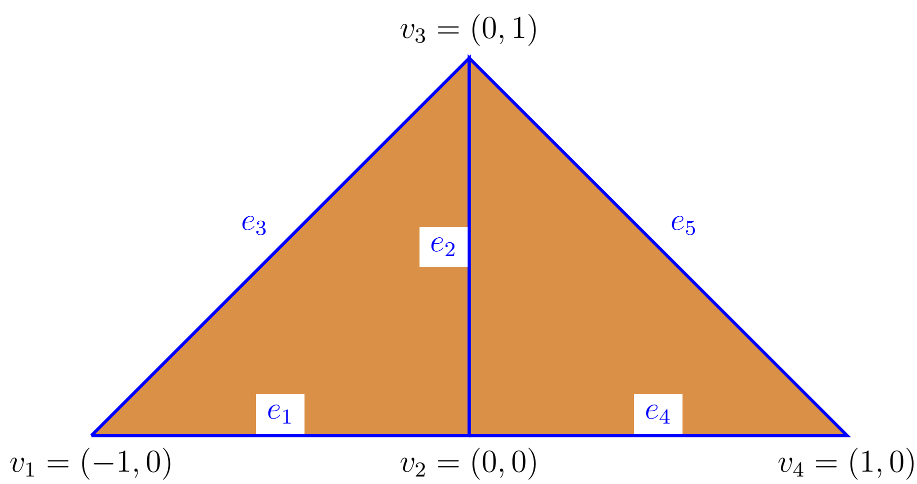


Figure 2: Mesh for sub-problem (3a)

(3b) (10 points) Now we consider a general polygon Ω equipped with a triangular mesh $\mathcal{M}$. Retain the notation $\boldsymbol{S}$ from sub-problem (3a) for the $N \times N$ matrix that transforms a coefficient vector w.r.t. to $\mathcal{B}_q$ into a coefficient vector w.r.t. $\mathcal{B}_h$.

Let $\mathsf{a}(\cdot,\cdot)$ be a continuous bilinear form on $H^1(\Omega)$, $\boldsymbol{A}_q$ and $\boldsymbol{A}_h$ be the Galerkin matrices of a w.r.t. to $\mathcal{B}_q$ and $\mathcal{B}_h$, respectively. How can $\boldsymbol{A}_h$ be obtained from $\boldsymbol{A}_q$ using the matrix $\boldsymbol{S}$?

(3c) (20 points) (Depends softly on (3a))

Given the mesh $\mathcal{M}$ in the usual LehrFEM data structure `mesh`, write an efficient function

```
function mu_hier = basistrf(Mesh, mu_nod),
```

where `mu_hier` and `mu_nod` correspond to the coefficient vectors $\vec{\boldsymbol{\mu}}_h$ and $\vec{\boldsymbol{\mu}}_q$, of a $u_N \in V_N$ with respect to $\mathcal{B}_h$ and $\mathcal{B}_q$, respectively. The argument `Mesh` passes a LehrFEM mesh data structure complete with edge information.

HINT: : You need not create the matrix $\boldsymbol{S}$!

Problem 4. Maximum principle [80 points]

For a constant $0 \leq c < 1$ we consider the 2nd-order elliptic boundary value problem

$$\begin{aligned} -(1-c)\triangle u + cu = f \quad & \text{in } \Omega \,, \\ u = 0 \quad & \text{on } \partial\Omega \,, \end{aligned} \tag{17}$$

where the source function $f \in L^2(\Omega)$ is continuous on Ω and satisfies $f(\boldsymbol{x}) \leq 0$ for all $\boldsymbol{x} \in \Omega$.

(4a) (5 points) State a quadratic minimization problem on a suitable function space, whose solution agrees with the solution of (17).

(4b) (15 points) (Depends softly on (4a))

Argue why

$$\max_{\boldsymbol{x}\in\Omega} u(\boldsymbol{x}) = 0 \tag{18}$$

holds. Why is $f \leq 0$ important for your argument?

HINT: Employ a reasoning similar to that in the "visual proof" of the maximum principle in the lecture, see figure 3.

(4c) (20 points) For the finite element Galerkin discretization of (17) on $\Omega =]0,1[^2$ we employ the linear Lagrangian finite element space $\mathcal{S}^0_{1,0}(\mathcal{M})$ on a "regular" mesh $\mathcal{M}$ as depicted in figure 4(a).

Write a MATLAB script

```
nomaxprinc
```

that runs a computational counterexample and produces a suitable output in order to demonstrate that for certain values of c the finite element solution $u_N \in \mathcal{S}^0_{1,0}(\mathcal{M})$ does not satisfy (18).

You may use the LehrFEM function (provided in the file `solveBVP.m`)

```
function u = solveBVP(M,c,f)
```

that computes the coefficient vector of a finite element solution $u_N \in \mathcal{S}^0_{1,0}(\mathcal{M})$ of the boundary value problem (17) with respect the nodal basis. It uses a "regular mesh" with $M \in \mathbb{N}$ cells in each direction as displayed in Fig. 4(a). The argument `c` passes $c \in [0,1[$, and `f` a handle of type `@(x1,x2)` to the source function f.

HINT: Try $c \approx 1$.

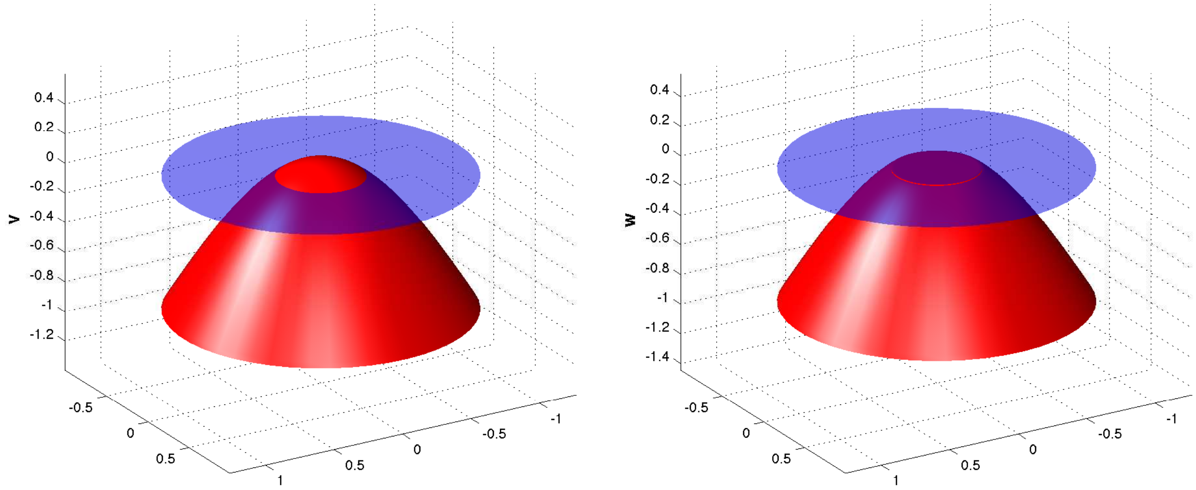


Figure 3: A visual cue for the proof requested in sub-problem (4b)

(4d) (20 points) Again, we consider the finite element Galerkin discretization of (17) on $\Omega =]0,1[^2$ based on $\mathcal{S}^0_{1,0}(\mathcal{M})$.

On a triangular mesh $\mathcal{M}$ as in Figure 4(a) with meshwidth $h := \frac{1}{M+1}$, $M \in \mathbb{N}$, we assemble the Galerkin matrix using the local numerical quadrature

$$\int_K \phi(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \approx \frac{1}{3}|K| \sum_{i=1}^{3} \phi(\boldsymbol{a}_i) \;, \quad K \in \mathcal{M} \;,$$

where $\boldsymbol{a}_i$ are the vertices of the triangle K.

Write a MATLAB function

```
function A = compA(M,c)
```

that computes the resulting (sparse!) Galerkin matrix, if the standard nodal basis of $\mathcal{S}^0_{1,0}(\mathcal{M})$ is used and the numbering of the basis functions is induced by the lexikographic numbering of the vertices as given in Figure 4(b).

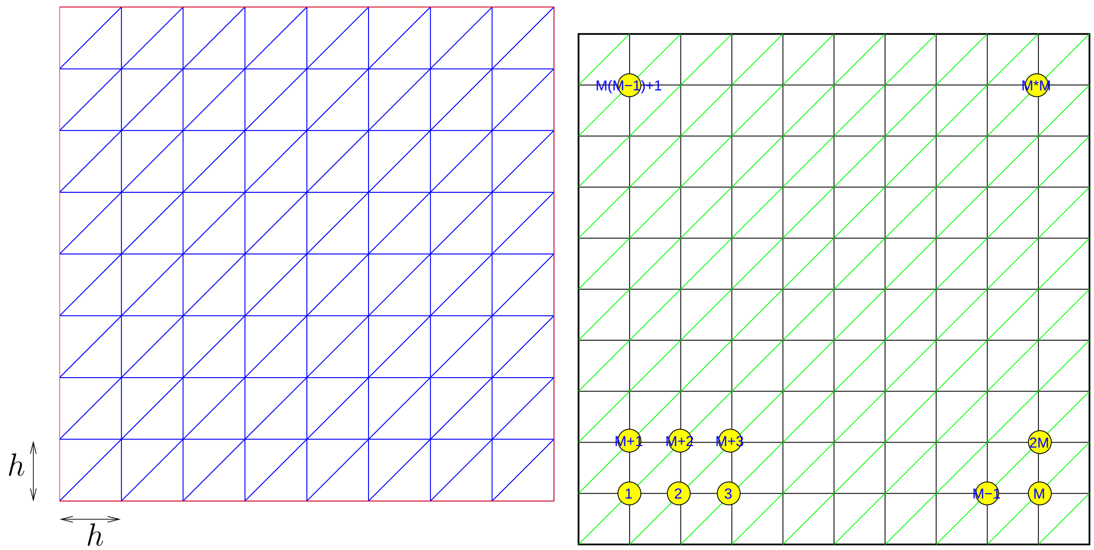
HINT: Several MATLAB commands come handy for this problem like `spdiags`, `kron`, `gallery('triadiag',...)`. A scrambled reference implementation of `compA` is provided in the file `compA_ref.p`.

(4e) (20 points) (Depends softly on (4d))

Show that for the Galerkin finite element discretization of (17) on $\Omega =]0,1[^2$ introduced in sub-problem (4d) the property $u_N(\boldsymbol{x}) \leq 0$ is satisfied for the finite element solution, if $f(\boldsymbol{x}) \leq 0$ for all $\boldsymbol{x} \in \Omega$.

Problem References

[NPDE] Lecture Slides for the course "Numerical Methods for Partial Differential Equations", SVN revision # 54024.



(a) Regular triangular mesh

(b) Lexikographic numbering of interior vertices

Figure 4: Mesh and vertex numbering for Problem 4

Last modified on April 11, 2013