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Spring Term 2014

Numerical Methods for Partial Differential Equations

ETH Zürich
D-MATH

Exam Summer 2014

The exam consists of 100 points + 10 bonus points.
This means that there are 110 total points but the mark is computed on a basis of 100 points.
Duration: 3 hours.
Slightly harder sub-problems are marked with an asterisk symbol *.

Problem 1 Stationary Reaction-Diffusion problem [50 points]

Consider the two-dimensional domain $\Omega = (0,1)^2$. Its boundary $\partial\Omega$ is divided into two parts: a Dirichlet boundary Γ_D and a Neumann boundary Γ_N, i.e. $\partial\Omega = \Gamma_D \cup \Gamma_N$, with $\Gamma_D = \{(x,0) \in \mathbb{R}^2, x \in (0,1)\} \cup \{(0,y) \in \mathbb{R}^2, y \in (0,1)\}$ and $\Gamma_N = \{(x,1) \in \mathbb{R}^2, x \in (0,1)\} \cup \{(1,y) \in \mathbb{R}^2, y \in (0,1)\}$. The domain is depicted in Fig. 1.1.

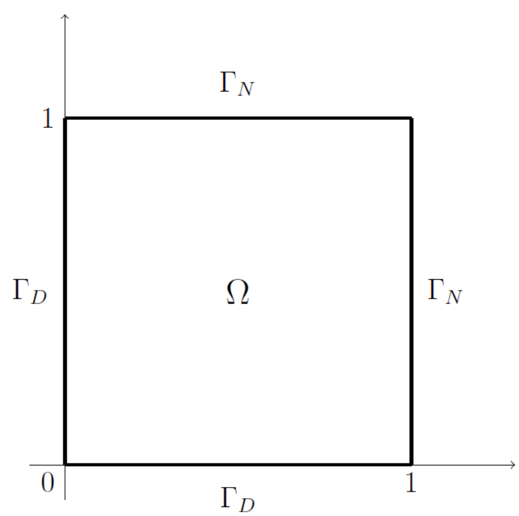

Figure 1.1: Domain for Problem 1.

On Ω we consider the PDE

$$-\Delta u(\boldsymbol{x}) + c(\boldsymbol{x})u(\boldsymbol{x}) = f(\boldsymbol{x}) \quad \boldsymbol{x} \in \Omega \tag{1.1}$$

$$u|_{\Gamma_D}(\boldsymbol{x}) = g(\boldsymbol{x}) \quad \boldsymbol{x} \in \Gamma_D \tag{1.2}$$

$$\left.\frac{\partial u}{\partial \boldsymbol{n}}\right|_{\Gamma_N}(\boldsymbol{x}) = h(\boldsymbol{x}) \quad \boldsymbol{x} \in \Gamma_N, \tag{1.3}$$

with $\boldsymbol{n}$ the outer unit normal to Γ_N, $f \in L^2(\Omega)$ and g, h continuous on $\partial\Omega$.
The coefficient $c = c(\boldsymbol{x})$ is *uniformly positive* and *bounded* in Ω.

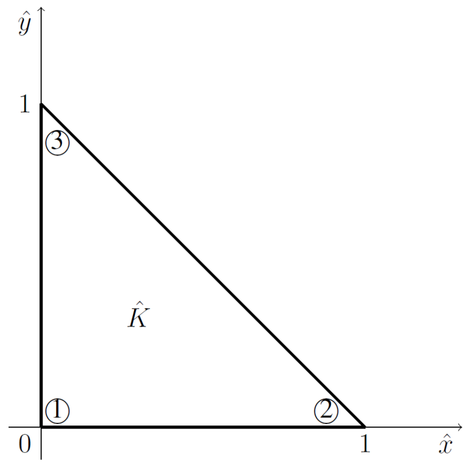

Figure 1.2: Reference triangle for subproblem (1f). The labels ①, ② and ③ denote the local numbering of the vertices.

(1a) Write the variational formulation for (1.1), specifying the bilinear form and linear form.

HINT: Don't forget to specify trial and test spaces.

(1b) Suppose now for this subtask that $h = 0$. Show that the solution to the variational formulation in (1a) exists and it is unique.

HINT: Use Lax-Milgram Lemma.

(1c) Specify where in subproblem (1b) you used that the coefficient $c = c(\boldsymbol{x})$ is uniformly positive and bounded.

(1d) * Consider now the general case that $h \neq 0$. Show that also in this case the solution exists and is unique.

HINT: For a function $v \in H^1(\Omega)$, the so-called *trace inequality* holds:

$$\|v\|^2_{L^2(\partial\Omega)} \leq C\|v\|_{L^2(\Omega)}\|v\|_{H^1(\Omega)}, \tag{1.4}$$

for a constant $C > 0$. In particular, this implies that

$$\|v\|_{L^2(\partial\Omega)} \leq \bar{C}\|v\|_{H^1(\Omega)}, \tag{1.5}$$

with $\bar{C} = \sqrt{C}$.

In the following we consider a discretization of the variational formulation in subproblem (1a) using *linear finite elements*.

(1e) Let $\hat{\lambda}_j = \hat{\lambda}_j(\hat{\boldsymbol{x}})$, $j = 1, 2, 3$, denote the shape functions for linear finite elements on the *reference element* (see Fig.1.2). Write down the analytical expression for $\hat{\lambda}_j = \hat{\lambda}_j(\hat{\boldsymbol{x}})$, $j = 1, 2, 3$.

(1f) Consider a constant coefficient $c = 1$.
We want to compute the element matrix $A_{\hat{K}}$ (i.e. the matrix associated to the bilinear form when restricted to an element) for the *reference triangle* $\hat{K}$ depicted in Fig. 1.2. The PDE (1.1) suggests that this matrix can be written as sum of two contributions: $A_{\hat{K}} = A_{\hat{K},1} + A_{\hat{K},2}$. Compute the entries of the matrices $A_{\hat{K},1}$ and $A_{\hat{K},2}$.

(1g) Consider now a generic triangle K with vertices $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$, considered as *column vectors* and numbered counterclockwise. The element matrix A_K for the triangle K is given by $A_K = A_{K,1} + A_{K,2}$, with the part $A_{K,1}$ associated to the diffusion term and the part $A_{K,2}$ associated to the reaction term. Suppose again a constant coefficient $c = 1$.
Using the shape functions $\left\{\hat{\lambda}_j\right\}_{j=1,2,3}$ and their gradients $\left\{\hat{\nabla}\hat{\lambda}_j\right\}_{j=1,2,3}$ *on the reference element*, give an expression for a generic entry $A_{K,1}(i,j)$ of $A_{K,1}$ and a generic entry $A_{K,2}(i,j)$ of $A_{K,2}$ ($i, j = 1, 2, 3$).

(1h) Consider $f = 0$ and a generic $h \neq 0$. Write the element load vector L_K (i.e. the vector associated to the linear form when restricted to an element) for a generic triangle K with vertices $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$, considered as *column vectors* and numbered counterclockwise.
Use the one-dimensional trapezoidal quadrature rule to compute the boundary integrals; express the element load vector in terms of the vertices $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$ and function values of h.

HINT: You may use the notation

$$\delta_{ij} = \begin{cases} 1 & \text{if } e_{ij} \in \Gamma_N \\ 0 & \text{otherwise,} \end{cases}$$

with e_{ij} denoting the edge connecting the vertices i and j, for $i, j = 1, 2, 3$.

(1i) We want to write the routine `assemLoad_LFE` to assemble the load vector associated to the linear form in subproblem (1a) in the case that $h = 0$ and $f \neq 0$.
Let `Mesh` be the mesh data stucture complete with vertex, edge, element and boundary edge information, let $\{\hat{\boldsymbol{x}}_i\}_{i=1}^N$ and $\{w_i\}_{i=1}^N$ be the nodes and weights for a quadrature formula on the reference triangle, and let F be the function handle to f.
Given `coords` the 3×2 matrix containing the vertex coordinates of a generic triangle K (i.e. `coords`$=\begin{pmatrix} \boldsymbol{a}_1^T \\ \boldsymbol{a}_2^T \\ \boldsymbol{a}_3^T \end{pmatrix}$, with $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ as in the previous subtasks), the routine call

$$L_{\text{loc}}=\texttt{LOAD\_LFE(coords,}\{\hat{\boldsymbol{x}}_i\}_{i=1}^N, \{w_i\}_{i=1}^N, F)$$

computes the element load vector for the triangle K.
The following pseudocode is provided:

`function` L=`assemLoad_LFE(Mesh,`$\{\boldsymbol{x}_i\}_{i=1}^N, \{w_i\}_{i=1}^N, F$)

for i=1:nElements **do**
    Extract global indices of element i $\rightarrow$ `vidx`
    Extract coordinates of vertices `vidx` $\rightarrow$ `coords`
    L_{loc}=`LOAD_LFE(coords,`$\{\hat{\boldsymbol{x}}_i\}_{i=1}^N, \{w_i\}_{i=1}^N, F$)
    for j=1:3 **do**
        L(`vidx`(j)) = $L_{\text{loc}}(j)$
    end for
end for

This pseudocode contains an (algorithmic) error.
Where is the error? How should it be corrected?

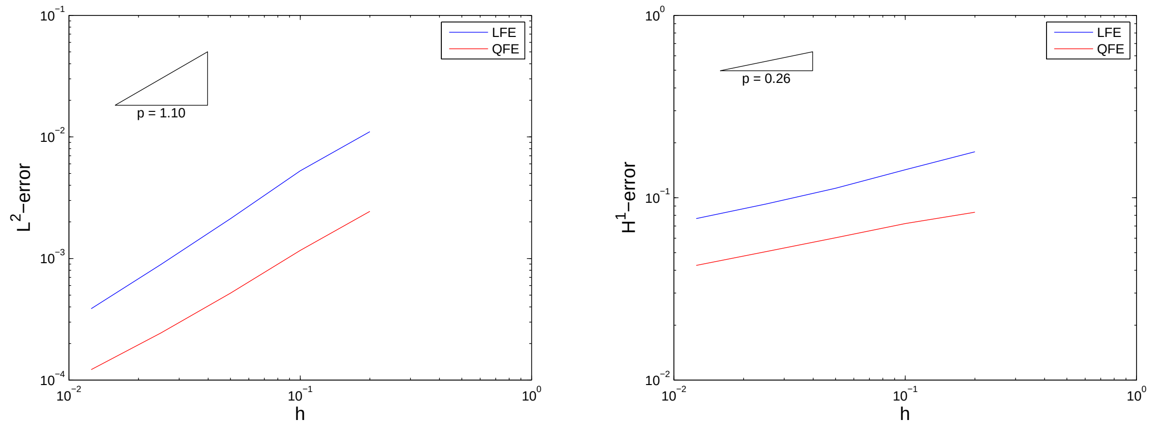

Figure 1.3: Convergence studies for subproblem (1l)

(1j) We want to compute the linear finite element solution u_N to the variational formulation of subproblem (1a) for the case that $g \neq 0$.
Denote by N_V the total number of mesh vertices and by N_{V,Γ_D} the number of vertices which are on the Dirichlet boundary Γ_D. Let $\boldsymbol{D}$ be the array of length $N_{V,D}$ containing the indices of the vertices which are on the Dirichlet boundary and **FreeDofs** the array of length $N_V - N_{V,D}$ containing the indices of all the other vertices, that is the vertices which are not on Γ_D.
We denote by $\boldsymbol{\mu} \in \mathbb{R}^{N_V}$ the *column* vector containing the coefficients of u_N with respect to the basis functions associated to all the mesh vertices. Then we have $\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_{\textbf{FreeDofs}} \\ \boldsymbol{\mu}_{\boldsymbol{D}} \end{pmatrix}$, with $\boldsymbol{\mu}_{\boldsymbol{D}} \in \mathbb{R}^{N_{V,D}}$ the coefficients with respect to the basis functions associated to nodes on Γ_D, and $\boldsymbol{\mu}_{\textbf{FreeDofs}} \in \mathbb{R}^{N_V - N_{V,D}}$ the coefficients with respect to the remaining basis functions.
Assume that you have already assembled the stiffness matrix $A \in \mathbb{R}^{N_V, N_V}$ and the load vector $L \in \mathbb{R}^{N_V}$ (considering all the mesh vertices as degrees of freedom).

(a) How should the vector $\boldsymbol{\mu}$ be initialized?

(b) How should the right-hand side L be modified in order to take into account the Dirichlet boundary conditions?

(c) Write down the algebraic system that has to be solved so that $\boldsymbol{\mu}$ contains the values of u_N in all the mesh vertices.

(1k) Suppose that the solution u to (1.1) is smooth (for example, $u \in C^\infty(\mathbb{R}^2)$). Which convergence orders for the L^2- and H^1- norms of the error do you expect when solving (1.1) with *linear* finite elements?
* And when solving (1.1) using *quadratic* finite elements?
State the convergence orders in terms of the meshwidth h.

(1l) We solve (1.1) with linear and quadratic finite elements for the case that the exact solution is $u_{\text{ex}}(\boldsymbol{x}) = (xy)^{\frac{3}{4}}$. The results of the convergence studies are shown in Fig. 1.3.

Why is there no gain in the convergence order when using quadratic finite elements?
Your statements must be supported by calculations.

(1m) * We compute the solution to (1.1) using linear finite elements in the case that the exact solution is $u_{\text{ex}}(\boldsymbol{x}) = x + y$. From the convergence study, we observe the following values:

h	L^2-error	H^1-error
0.2500	4.2043e-16	1.31832e-15
0.1250	9.47152e-16	3.34797e-15
0.0625	3.27805e-15	9.54326e-15
0.0312	4.12217e-15	1.68383e-14

Give a motivation for why the error is of the order of the machine precision for all the meshwidths considered.

HINT: Consider the functional space to which u_{ex} belongs.

Problem 2 Stability property for the Poisson equation [10 points]

The aim of this problem is to better understand what does the *stability* mean for a differential equation. We consider the Poisson equation with homogeneous Dirichlet boundary conditions:

$$\begin{aligned} -u''(x) &= f(x), \quad \forall x \in \Omega = (0,1) \\ u(0) &= u(1) = 0. \end{aligned} \tag{2.1}$$

for $f \in C([0,1])$.

In applications the exact source term $f(x)$ is not available. What is available is some perturbation of it

$$\tilde{f}(x) = f(x) + \eta(x), \tag{2.2}$$

where $\eta(x)$ is some noise introduced, for example, by some measurement error.
Then, what can be actually computed is the solution $\tilde{u}$ to the perturbed system

$$\begin{aligned} -\tilde{u}''(x) &= \tilde{f}(x), \quad \forall x \in \Omega = (0,1) \\ \tilde{u}(0) &= \tilde{u}(1) = 0. \end{aligned} \tag{2.3}$$

(2a) Show that

$$\|u - \tilde{u}\|_\infty \le \frac{1}{8}\|f - \tilde{f}\|_\infty. \tag{2.4}$$

HINT: From the lecture you know that any solution v to the Poisson equation (2.1) with right-handside g satisfies the estimate

$$\|v\|_\infty \le \frac{1}{8}\|g\|_\infty.$$

(2b) Consider now that we want to compute the solution to (2.3) numerically and denote by $\tilde{u}_h$ the discrete solution. We are interested in estimating how well $\tilde{u}_h$ approximates the exact solution u to the *unperturbed* problem (2.3).
Show that the following estimate holds:

$$\|u - \tilde{u}_h\|_\infty \le \frac{1}{8}\|\eta\|_\infty + \|\tilde{u} - \tilde{u}_h\|_\infty. \tag{2.5}$$

Note that here we are not making any assumption on the discretization scheme.

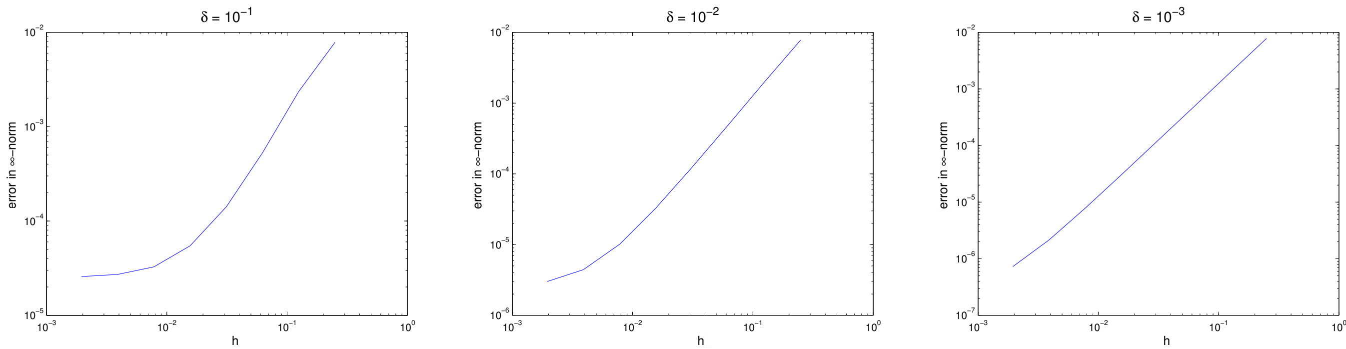

Figure 2.1: Plots for subproblem (2c).

(2c) Suppose that $\eta(x) = \delta \sin(2\pi x)$, for some $\delta > 0$, so that

$$\tilde{f} = f + \delta \sin(20\pi x). \tag{2.6}$$

Let us consider $\delta = 10^{-1}, 10^{-2}, 10^{-3}$ and suppose that for each of these values we made a convergence study for $\tilde{u}_h$ considering the error $\|u - \tilde{u}_h\|_\infty$. The convergence plots in doubly logarithmic scale (loglog) are shown in Fig. 2.1.
Observe and compare the plots and comment the following questions:

- Why is there a plateau (flat region) for small mesh size h?
- How does the plateau change with δ? Why?

Problem 3 Finite difference method for the heat equation [15 points]

Let $\Omega := (0, 1)$ and consider a one-dimensional *heat equation* (an example of a parabolic PDE) with homogeneous Dirichlet boundary conditions:

$$\frac{\partial}{\partial t} u(x,t) - \frac{\partial^2}{\partial x^2} u(x,t) = 0 \qquad \text{in } (0,T] \times \Omega, \tag{3.1}$$
$$u(x,t) = 0 \qquad \text{on } (0,T] \times \partial\Omega, \tag{3.2}$$
$$u(x,0) = u_0(x) \qquad \text{on } \{0\} \times \Omega. \tag{3.3}$$

(3a) Let $E(t)$ denote the energy of the solution $u(x,t)$ to (3.1) at time t, i.e.

$$E(t) := \frac{1}{2} \int_\Omega (u(x,t))^2 dx. \tag{3.4}$$

Prove, that solution $u(x,t)$ to the above heat equation (3.1) satisfies the energy inequality, i.e.

$$E(t) \leq E(0). \tag{3.5}$$

HINT: Differentiate $E(t)$ with respect to time t and then use (3.1) and integration by parts.

Next, we aim to discretize the above heat equation using central finite differences coupled with an explicit time-stepping scheme.

To discretize the spatial domain $\Omega = (0,1)$, we subdivide the interval $[0,1]$ in $N+1$ subintervals using equispaced grid points $\{x_0 = 0, x_1, \ldots, x_N, x_{N+1} = 1\}$. Let us denote by $h = |x_1 - x_0|$ the mesh size. To discretize the time domain $[0,T]$, consider $M \in \mathbb{N}$ equispaced time points

$$t_0 = 0, \quad t_1, \quad \ldots, \quad t_M = T,$$

and denote the time step size by $\Delta t := |t_{n+1} - t_n| = T/M$. Then, at a time point $t = t_n$ with $n = 0, \ldots, M-1$, the solution $u(x_j, t^n)$ at point x_j and at time t^n is denoted by U_j^n.

Consider the following discretization scheme:

$$U_j^{n+1} = U_j^n - \Delta t \frac{U_{j-1}^n - 2U_j^n + U_{j+1}^n}{h^2}, \quad \text{with} \quad \Delta t = h. \tag{3.6}$$

(3b) Show, that (3.6) is a consistent discretization of (3.1), i.e. that as $h \to 0$, scheme (3.6) approximates the continuous problem (3.1).

(3c) It is known that the scheme (3.6) produces solutions which blow up, i.e. the discrete version of the energy inequality (3.5)

$$E^n \leq E^0 \tag{3.7}$$

with

$$E^n = \frac{1}{2} \sum_{j=1}^{N} (U_j^n)^2 \tag{3.8}$$

is violated. What is the reason for this blow up? What needs to be changed (no proof required) in the discretization scheme (3.6) such that the discrete energy inequality (3.7) holds?

Problem 4 Transport in 1-D and Upwinded Finite Differences [15 points]

Consider the one-dimensional linear transport equation:

$$\begin{aligned} \frac{\partial}{\partial t} u(x,t) + a(x) \frac{\partial}{\partial x} u(x,t) &= 0, \qquad \forall (x,t) \in \mathbb{R} \times \mathbb{R}_+, \\ u(x,0) &= u_0(x), \qquad \forall x \in \mathbb{R}, \end{aligned} \tag{4.1}$$

with coefficient $a(x)$ given by

$$a(x) = \begin{cases} 0 & \text{if } x < 0, \\ x & \text{if } 0 \leq x < 1, \\ 1 & \text{if } 1 \leq x. \end{cases} \tag{4.2}$$

(4a) * Write down the equation of characteristics of (4.1) and sketch characteristic curves in the sub-domain $[-1, 2]$ up to time $t = 1$. Use the equation of characteristics to derive an expression for the exact solution of (4.1) in terms of initial data $u_0(x)$.

HINT: For the characteristics equation, do not forget to specify the initial conditions.

HINT: The solution to ODE of the form $\frac{\partial x(t)}{\partial t} = \alpha x(t)$ is given by $x(t) = x(0)\exp(\alpha t)$ for $\alpha \neq 0$.

HINT: To find the characteristic curves with starting point $0 < x_0 < 1$ and to find the solution $u(x,t)$ with $x > 0$ and $1 + t > x$, you will need to carefully analyze what happens at $x = 1$.

(4b) Write down the *upwind* Finite Difference scheme, which approximates the solution $u(x,t)$ in (4.1) at points $U_j^n = u(x_j, t^n)$ with $j \in \mathbb{Z}$ and $n \in \mathbb{N}$. Here, $\ldots, x_{j-1}, x_j, x_{j+1}, \ldots$ denote the mesh points (with a given *uniform* mesh width Δx) and $0 = t^0, t^1, \ldots$ denote the time instances. Also, specify under which conditions on the mesh size Δx and the time step size Δt is the upwind Finite Difference scheme stable.

Problem 5 Entropy Solutions for a Scalar Conservation Law [20 points]

We consider the Cauchy problem on $\mathbb{R} \times (0,T)$ for the scalar conservation law

$$\frac{\partial}{\partial t} u(x,t) + \frac{\partial}{\partial x} f(u) = 0. \tag{5.1}$$

with the flux $f(u) = (u^3)/3$ and with initial condition $u(x,0) = u_0(x)$, $x \in \mathbb{R}$, satisfying

$$0 \le u_0(x) \le 1, \quad x \in \mathbb{R}. \tag{5.2}$$

(5a) Most of the examples in the lecture slides are for **convex** flux functions!
Determine the following:

- Is the flux function $f(u) = (u^3)/3$ **convex** for all values of $u \in \mathbb{R}$?
- Is the flux function $f(u) = (u^3)/3$ **convex** for all values of $u \in [0,1]$?

Is the convexity of f in the interval $[0,1]$ sufficient if we consider only initial data given in (5.2)? Explain why.

(5b) Determine the entropy solutions of the Riemann problem with

$$\text{(i)} \quad u_0(x) = \begin{cases} 0 & \text{for } x \le 0, \\ 1 & \text{for } x > 0, \end{cases} \qquad \text{(ii)} \quad u_0(x) = \begin{cases} 1 & \text{for } x \le 0, \\ 0 & \text{for } x > 0. \end{cases}$$

(5c) Now consider (5.1) in the domain $D = (-1,3) \subset \mathbb{R}$, with the flux function $f(u) = (u^2)/2$ and the following initial conditions:

$$u(x,0) = \begin{cases} -1, & \text{if} \quad 0 \le x, \\ 1, & \text{if} \quad 0 < x < 1, \\ 0, & \text{otherwise}. \end{cases} \tag{5.3}$$

Three different numerical fluxes - Godunov, Roe and Lax-Friedrichs - were used to approximate the solutions to (5.1) with (5.3) and "outflow" boundary conditions. The results at time $T = 0.5$ for 100 mesh cells and the convergence plots are depicted in Figure 5.1, where the solvers are labeled using letters (a), (b), and (c).

Which letter corresponds to which numerical flux?
Explain your answer.

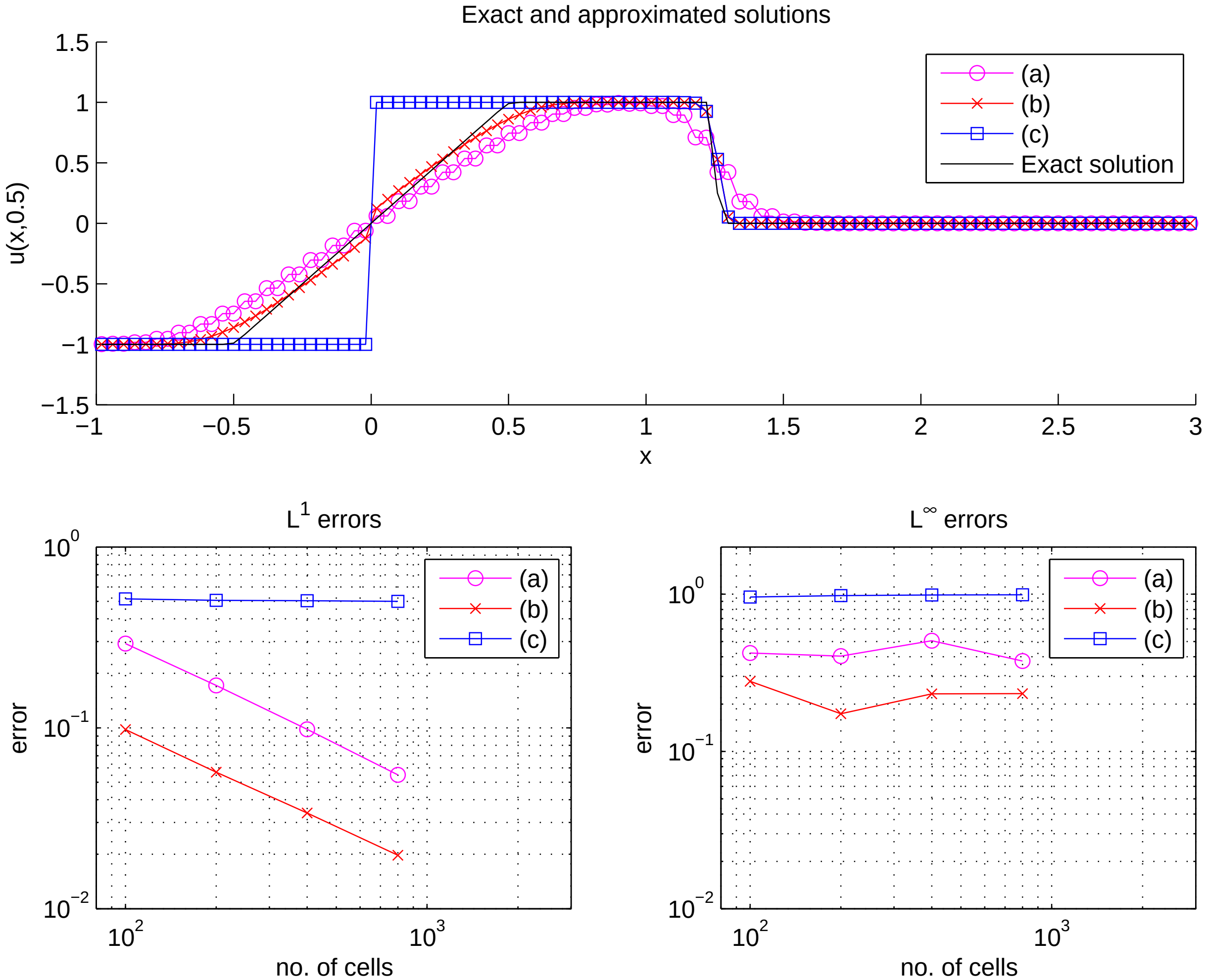

Figure 5.1: Exact solutions and FVM approximations for the solution of (5.1) with (5.3).