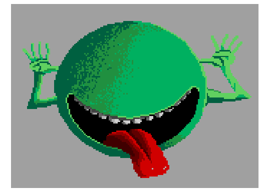
Course 401-3663-00L: Numerical Methods for Partial Differential Equations
Examination, Spring 2012

Prof. Philipp Grohs



Dont't panic !
Good luck !

Duration of examination: 180 minutes

Problem 1. (CFL-condition)

We consider the Cauchy problem for the one-dimensional linear advection equation ($c \in \mathbb{R}$)

$$\begin{aligned} \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} &= 0 \quad \text{on } \mathbb{R} \times [0,T] \ , \\ u(x,0) &= u_0(x) \quad \forall x \in \mathbb{R} \ . \end{aligned} \tag{1}$$

We compute approximations $\mu_j^{(k)} \approx u(jh, k\tau)$ on an equidistant space-time mesh with spatial meshwidth $h > 0$ and uniform timestep $\tau > 0$ by means of the discrete evolution

$$\mu_j^{(k)} = \mu_j^{(k-1)} - c\gamma(\mu_j^{(k-1)} - \mu_{j-1}^{(k-1)}) - \tfrac{1}{2}c\gamma(1-c\gamma)(\mu_j^{(k-1)} - 2\mu_{j-1}^{(k-1)} + \mu_{j-2}^{(k-1)}) \ , \tag{2}$$

with $\gamma := \frac{\tau}{h}$, $k \in \mathbb{N}$, $j \in \mathbb{Z}$, with initial values $\mu_j^{(0)} := u_0(jh)$, $j \in \mathbb{Z}$.

(1a) (2 points) What is a Cauchy problem?

(1b) (5 points) Using the notation $\vec{\boldsymbol{\mu}}^{(k)} := (\mu_j^{(k)})_{j\in\mathbb{Z}}$, (2) is equivalent to

$$\vec{\boldsymbol{\mu}}^{(k)} = \boldsymbol{\mathcal{L}}(\vec{\boldsymbol{\mu}}^{(k-1)}) \ , \tag{3}$$

with a linear operator

$$\boldsymbol{\mathcal{L}} : \mathbb{C}^{\mathbb{Z}} \mapsto \mathbb{C}^{\mathbb{Z}} \quad , \quad (\boldsymbol{\mathcal{L}}\vec{\boldsymbol{\mu}})_j := \sum_{\ell=-2}^{0} c_\ell \mu_{j+\ell} \ , \quad j \in \mathbb{Z} \ , \tag{4}$$

with suitable $c_{-2}, c_{-1}, c_0 \in \mathbb{R}$. Determine the coefficients c_{-2}, c_{-1}, c_0.

(1c) (15 points) For given $\xi \in \mathbb{C}$, $c \in \mathbb{R}$, determine $\lambda(\xi) \in \mathbb{C}$ such that

$$\boldsymbol{\mathcal{L}}\left((\exp(\imath\xi j))_{j\in\mathbb{Z}}\right) = \lambda(\xi)(\exp(\imath\xi j))_{j\in\mathbb{Z}} \ . \tag{5}$$

Hint: $\imath$ stands for the imaginary unit, $\imath^2 = -1$. Use the fundamental property of the exponential function $\exp(a+b) = \exp(a)\exp(b)$.

(1d) (5 points) Compute analytically the solution of the discrete evolution (2) for $u_0(x) = \exp(\imath\xi x)$, $\xi \in \mathbb{C}$.

Hint: Use (5).

(1e) (5 points) Show that for $c\tau = 2.1h$ solutions of (2) can suffer a blow-up

$$\lim_{k\to\infty} \max_{j\in\mathbb{Z}} u_j^{(k)} = \infty \; . \tag{6}$$

Hint: Plot $|\lambda(\xi)|$ for $-\pi \leq \xi \leq \pi$.

(1f) (10 points) Give a (geometric) argument why the CFL condition $c\tau \leq 2h$ is *necessary* to have convergence $\mu_N^{(N)} \to u(1,\beta)$, when $\beta := \frac{\tau}{h}$ is kept fixed and $h := \frac{1}{N} \to 0$, $N \in \mathbb{N}$.

Problem 2. (Radiative cooling)

The evolution of the temperature distribution $u = u(\boldsymbol{x})$ in a homogeneous "2D body" (occupying the space $\Omega \subset \mathbb{R}^2$) with radiative cooling is modelled by the linear parabolic intial-boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u = 0 \quad &\text{in } \Omega \times [0,T] \; , \\ -\operatorname{\mathbf{grad}} u \cdot n = \gamma u \quad &\text{on } \partial\Omega \times [0,T] \; , \\ u(\boldsymbol{x},0) = u_0(\boldsymbol{x}) \quad &\text{in } \Omega \; , \end{aligned} \tag{7}$$

with $\gamma > 0$.

(2a) (13 points) Derive the spatial variational formulation for (7). Do not forget to specify the function spaces. Compute exactly the mass matrix and the Galerkin matrix for the unit triangle (with vertices $(0,0)$, $(1,0)$ and $(0,1)$). Assume that the x-axis forms part of $\partial\Omega$.

(2b) (15 points) For the spatial Galerkin discretization of (7) we employ linear finite elements on a triangular mesh $\mathcal{M}$ of Ω (FE space $\mathcal{S}_1^0(\mathcal{M})$) with polygonal boundary approximation. All integrals are evaluated by local vertex based quadrature formulas (the trapezoidal rule).

Implement a MATLAB function

```
[M,A] = getMatLFE(mesh,gamma)
```

that computes the matrices $\mathbf{M}, \mathbf{A} \in \mathbb{R}^{N,N}$ for the semi-discrete evolution

$$\mathbf{M}\frac{d}{dt}\vec{\boldsymbol{\mu}}(t) + \mathbf{A}\vec{\boldsymbol{\mu}}(t) = 0 \tag{8}$$

resulting from the finite element discretization of (7), when standard nodal bassis functions are used. Here, the argument `mesh` passes a LehrFEM mesh data structure, and `gamma` supplies the value of γ.

To facilitate the implementation, a MATLAB template is provided in `getMatLFE.m` which already computes the stiffness matrix for $-\Delta$. A reference file is provided as `getMatLFERef.p`. The flag for Robin boundary conditions is `-1`.

(2c) (20 points) Implement a MATLAB function

```
mufinal = RadTEvl(u0,gamma,mesh,m)
```

that carries out m uniform timesteps of the L-stable SDIRK-2 implicit 2-stage Runge-Kutta method with Butcher scheme

$$\begin{array}{c|cc} \lambda & \lambda & 0 \\ 1 & 1-\lambda & \lambda \\ \hline & 1-\lambda & \lambda \end{array} \qquad \lambda := 1 - \tfrac{1}{2}\sqrt{2}\,, \tag{9}$$

in order to solve (7) over the time interval $[0, 1]$. The finite element Galerkin discretization from sub-problem (2b) is used in space. The argument `u0` is a column vector that passes the values of the initial temperature distribution in the vertices of the mesh. The return value provides the basis coefficients of the approximation of $u(\cdot, 1)$. This function will be called within the driver routine `driver_evl.m`.

A MATLAB template is provided in `RadTEvl.m`, and a reference implementation in `RadTEvlRef.p`.

(2d) (10 points) Write a MATLAB function

```
avg = lfeavg(u,mesh)
```

that computes $\int_\Omega u \,\mathrm{d}\boldsymbol{x}$ for $u \in \mathcal{S}_1^0(\mathcal{M})$. The argument `u` passes the coefficients of u w.r.t. the standard nodal basis of $\mathcal{S}_1^0(\mathcal{M})$, while `mesh` contains a LehrFEM mesh data structure.

Hint. A reference implementation is provided as `lfeavgRef` (File `lfeavgRef.p`).

(2e) (8 points) For the evolution problem (7) on $\Omega =]0, 1[^2$ track the behavior of the thermal energy

$$E(t) = \int_\Omega u(\boldsymbol{x}, t) \,\mathrm{d}\boldsymbol{x} \tag{10}$$

over the period $[0, T]$ for $u_0 \equiv 1$, $\gamma = 1$. Use the fully discrete evolution implemented in `RadTEvl` and extend it to

```
[mufinal,E] = RadTEvl(u0,gamma,mesh,m) ,
```

where `E` returns approximations for $E(t_k)$ for $k = 0, \dots, m$ (t_k are the points of the equidistant temporal grid).

Extend the MATLAB script `driver_evl.m` to plot the approximation for $E(t)$ that you have computed as a function of t for $m = 100$ and the mesh supplied in the file `evlmesh.mat`.

Hint: The plot you should get is depicted in Figure 1.

Problem 3. (Best approximation in L^2)

Given a triangular mesh $\mathcal{M}$ of $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, we consider the minimization problem

$$u_N = \underset{v_N \in \mathcal{S}_1^0(\mathcal{M})}{\operatorname{argmin}} \int_\Omega |f - v_N|^2 \,\mathrm{d}\boldsymbol{x}\ , \tag{11}$$

where $f \in L^2(\Omega)$.

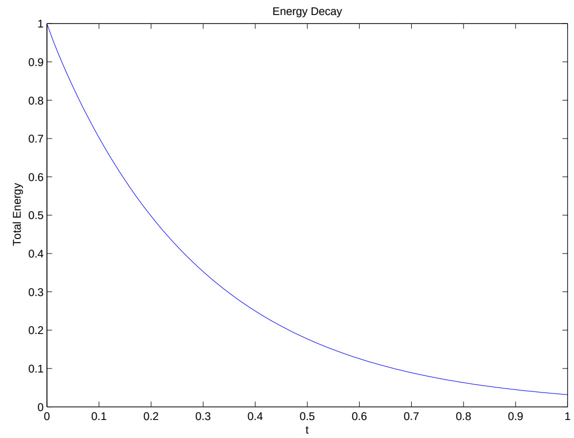


Figure 1: Result of sub-problem (2e)

(3a) (8 points) Show that (11) is a minimization problem for a quadratic functional of the form $J(v) = \frac{1}{2}\mathsf{a}(v,v) - \ell(v) + c$ with a bilinear form a and a linear form ℓ and $c \in \mathbb{R}$. Write down the concrete expressions for a and ℓ in the case of (11).

(3b) (4 points) Argue why the minimization problem (12) has a unique solution.

(3c) (8 points) State the linear variational problem that is equivalent to (11). Do not forget to specify the right function spaces.

(3d) (4 points) Now and for all sub-problems below we always consider $\Omega =]0,1[$, that is, $d = 1$, and an equidistant mesh $\mathcal{M}$ with meshwidth $h = \frac{1}{N}$, $N \in \mathbb{N}$. What is the dimension of $\mathcal{S}_1^0(\mathcal{M})$ is this case?

(3e) (10 points) In the setting of (3d), compute the Galerkin matrix for the linear variational problem from (3c), when the standard global shape functions ("tent functions") of $\mathcal{S}_1^0(\mathcal{M})$ are used and numbered from left to right.

(3f) (15 points) Assume the setting of (3d). Write a MATLAB routine

```
phi = rhsL2(xi,N)
```

that computes the right hand side vector for the discrete variational problem from (3c) for

$$f(x) = \begin{cases} 1 & \text{for } \xi < x < 1 \ , \\ 0 & \text{elsewhere} \end{cases} \quad , \quad 0 < \xi < 1 \ . \tag{12}$$

Of course, the standard "tent function" basis of $\mathcal{S}_1^0(\mathcal{M})$ is to be used with numbering from left to right.

Hint. For a reference implementation refer to `rhsL2Ref`.

(3g) (10 points) Write a MATLAB function

```
u = get_best_app(xi,N)
```

that solves (11) for f from (12) and returns the coefficients of u_N with respect to the “tent function” basis.

Hint. A reference implementation is available as `get_best_appRef`.

(3h) (10 points) Implement a MATLAB function

```
diff = l2dist(xi,uN,N)
```

that computes $\|f - u_N\|_{L^2(\Omega)}$ for f from (12) (characterized by parameter ξ) and $u_N \in \mathcal{S}_1^0(\mathcal{M})$ described by its basis coefficients in the vector `uN`.

Hint. A reference implementation is supplied by `l2distRef`. The integral of the square of a linear function that takes values V and $H\hat{\text{A}}$ at the endpoints of an interval of length L is $L(V^2 + H^2 + (V + H)^2)$. It is useful to do this computation in an auxiliary function.

(3i) (15 points) For $\xi = \frac{1}{2}\sqrt{2}$ choose f according to (12). Then, for $N = 10, 20, 40, 80, 160, 320, 640$ compute

$$\min_{v_N \in \mathcal{S}_1^0(\mathcal{M})} \int_\Omega |f - v_N|^2 \,\mathrm{d}\boldsymbol{x} \ ,$$

and plot its square root versus N in a suitable scale. Based on your observations characterize qualitatively and quantitively the N-asymptotic behavior of the $L^2(\Omega)$-best approximation error of $\mathcal{S}_1^0(\mathcal{M})$ for that particular f.

References

[NPDE] Lecture slides for course ”Numerical Methods for Partial Differential Equations”, Subversion Revision