Algebraic Geometry
D-MATH, FS 2016

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Exercise Sheet 10

1. a) Make the following statement precise and prove it:
"A hypersurface H in $\mathbb{P}^n$ of degree d meets a line l not contained in H in d points, counted with multiplicity."

b) Hypersurfaces of degree d in $\mathbb{P}^n$ are parametrized by $\mathbb{P}(\mathbb{C}[x_0, \ldots, x_n]_d)$ (see Exercise 6, Sheet 2) and lines $l \subset \mathbb{P}^n$ are parametrized by $\mathrm{Gr}(2, n+1)$. Show that the set

$$S = \{([f], l) : l \text{ intersects } V(f) \text{ in } d \text{ distinct points}\}$$

contains a nonempty Zariski-open subset of the variety $\mathbb{P}(\mathbb{C}[x_0, \ldots, x_n]_d) \times \mathrm{Gr}(2, n+1)$. We say that a line and a degree d hypersurface *in general position* intersect in exactly d points.

2. In this exercise, we want to show the following result.

Theorem 1. *Let $C \subset \mathbb{P}^2$ be an irreducible curve of degree d. Then C has at most $\binom{d-1}{2}$ singular points.*

a) Show the Theorem for $d = 1$.

b) For $d = 2$ recall the following result from Sheet 4, Exercise 4:

Lemma 1. *For five points in general linear position in $\mathbb{P}^2$ there exists a rational normal curve $\nu : \mathbb{P}^1 \to \mathbb{P}^2$ passing through them.*

Use this together with the Theorem of Bezout to show that every irreducible conic C in the plane $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$, hence has no singular point as desired. *Note*: This result can also be shown by projection from a point $p \in C$ to a line $l \subset \mathbb{P}^2$ not going through p.

c) Show the Theorem for $d \geq 3$ as follows: assume we have distinct singular points $a_1, \ldots, a_{\binom{d-1}{2}+1}$ of C. Choose additional points $b_i \in C$ and construct a curve C' of degree d' going through all points a_j, b_i. Arrive at a contradiction using Bezout's Theorem. To construct C' use Exercise 6, Sheet 2.

d) Show that a (not necessarily irreducible) curve C in $\mathbb{P}^2$ of degree d has at most $\binom{d}{2}$ singular points. Can you find an example for each d where this number is reached?

Due May 20.