Exercise Sheet 11

1. Define an affine scheme $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$ and show that

a) the stalk at $p \in \mathrm{Spec}(A)$ is A_p, the localization of A along p.

b) for f be an element of A,

$$\mathcal{O}_{\mathrm{Spec}(A)}(D(f)) = A_f.$$

Here $D(f)$ is the open set of primes not containing f and A_f is the localization of A at the element f. In particular $\mathcal{O}_{\mathrm{Spec}(A)}(\mathrm{Spec}(A)) = A$.

2. Prove that the morphisms from $\mathrm{Spec}(B)$ to $\mathrm{Spec}(A)$ as locally ringed spaces are in bijective correspondence to the ring homomorphisms $A \to B$.

3. Let $\mathcal{F}$ be a presheaf on a topological space X. For every $x \in X$, let $\mathcal{F}_x$ be the stalk of $\mathcal{F}$ at x. For $x \in U$, U open, let $\rho_{U,x} : \mathcal{F}(U) \to \mathcal{F}_x$ be the induced restriction map. Define a sheaf $\mathcal{F}^{\mathrm{sh}}$, the sheafification of $\mathcal{F}$ or the sheaf *associated to* $\mathcal{F}$, by

$$\mathcal{F}^{\mathrm{sh}}(U) = \{(s_x \in \mathcal{F}_x)_{x \in U} \mid \text{ for all } x \in U \text{ there exists an open set } V \text{ with } x \in V \subset U \text{ and } s \in \mathcal{F}(V), \text{ such that } \rho_{V,y}(s) = s_y \text{ for all } y \in V\}$$

(i) Show that $\mathcal{F}^{\mathrm{sh}}$ is a sheaf.

(ii) Prove that $(\mathcal{F}^{\mathrm{sh}})_x \cong \mathcal{F}_x$ for all $x \in X$.

(iii) Let $f : \mathcal{F} \to \mathcal{F}^{\mathrm{sh}}$ be the natural map given by $s \in \mathcal{F}(U) \mapsto (\rho_{U,x}(s))_{x \in U} \in \mathcal{F}^{\mathrm{sh}}(U)$. Show that it satisfies the following universal property: For any sheaf $\mathcal{G}$ and any map of presheaves $g : \mathcal{F} \to \mathcal{G}$, there exists a unique map $\bar{g} : \mathcal{F}^{\mathrm{sh}} \to \mathcal{G}$ such that $\bar{g} \circ f = g$.

Due May 27.