Algebraic Geometry

D-MATH, FS 2016

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Exercise Sheet 12

1. Given a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of abelian groups on a space X and $U \subset X$ open, let

$$\ker^{\mathrm{pre}}(\varphi)(U) = \ker(\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)),$$
$$\mathrm{coker}^{\mathrm{pre}}(\varphi)(U) = \mathrm{coker}(\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)).$$

a) Describe the natural "restriction maps"

$$\ker^{\mathrm{pre}}(\varphi)(U) \to \ker^{\mathrm{pre}}(\varphi)(V), \mathrm{coker}^{\mathrm{pre}}(\varphi)(U) \to \mathrm{coker}^{\mathrm{pre}}(\varphi)(V)$$

for $V \subset U$ open and show that this data defines presheaves of abelian groups.

b) Prove that for the stalks of the above presheaves we have

$$\ker^{\mathrm{pre}}(\varphi)_p = \ker(\mathcal{F}_p \xrightarrow{\varphi_p} \mathcal{G}_p), \mathrm{coker}^{\mathrm{pre}}(\varphi)_p = \mathrm{coker}(\mathcal{F}_p \xrightarrow{\varphi_p} \mathcal{G}_p).$$

c) Show that $\ker(\varphi) = \ker^{\mathrm{pre}}(\varphi)$ is a sheaf if $\mathcal{F}, \mathcal{G}$ are sheaves.

d) For $X = \mathbb{C}$ let $\mathcal{F} = (\mathcal{O}, +)$ be the sheaf of holomorphic functions (with addition) and $\mathcal{G} = (\mathcal{O}^*, \cdot)$ be the sheaf of nowhere zero holomorphic functions (with multiplication). Then there is a map $\exp : \mathcal{O} \to \mathcal{O}^*$ of sheaves of abelian groups defined by

$$\exp(U) : \mathcal{O}(U) \to \mathcal{O}^*(U), f \mapsto \exp(f).$$

i) Compute $\ker(\exp)$.

ii) Show that $\mathrm{coker}^{\mathrm{pre}}(\exp)$ is not a sheaf.

iii) In general, for a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of sheaves of abelian groups, we define the cokernel of φ as the sheafification

$$\mathrm{coker}(\varphi) = (\mathrm{coker}^{\mathrm{pre}}(\varphi))^{\mathrm{sh}}$$

of $\mathrm{coker}^{\mathrm{pre}}(\varphi)$. Compute $\mathrm{coker}(\exp)$.

2. a) Let $\mathcal{F}, \mathcal{G}$ be sheaves on a topological space X. Show that a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ is an isomorphism if and only if the induced maps $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ on the stalks at all points $p \in X$ are isomorphisms.

b) Let $\mathcal{F}, \mathcal{F}', \mathcal{F}''$ be sheaves of abelian groups on X and assume we have a sequence

$$\mathcal{F}' \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}''$$

such that $\psi \circ \varphi = 0$. The sequence is called *exact at* $\mathcal{F}$ if the natural map

$$\operatorname{im}(\varphi) = \ker(\mathcal{F} \to \operatorname{coker}(\varphi)) \to \ker(\psi)$$

is an isomorphism. Show that this is equivalent to the condition that the sequence

$$\mathcal{F}'_p \xrightarrow{\varphi_p} \mathcal{F}_p \xrightarrow{\psi_p} \mathcal{F}''_p$$

of maps induced on the stalks is exact for all $p \in X$.

3. Let X be a topological space and let $\mathcal{U} = (U_\alpha)_{\alpha \in A}$ be a base of the topology of X. A *sheaf* F *on the base* $\mathcal{U}$ is a collection $(F(U_\alpha))_{\alpha \in A}$ of sets together with morphisms

$$\rho_{\beta\alpha} : F(U_\alpha) \to F(U_\beta)$$

for $U_\beta \subset U_\alpha$, such that $\rho_{\alpha\alpha} = \mathrm{id}$ and

$$\rho_{\gamma\beta} \circ \rho_{\beta\alpha} = \rho_{\gamma\alpha}$$

for $U_\gamma \subset U_\beta \subset U_\alpha$. Moreover, for $U_\alpha = \bigcup_{\beta \in B} U_\beta$ and elements $f_\beta \in F(U_\beta)$ such that $\rho_{\gamma\beta}(f_\beta) = \rho_{\gamma\beta'}(f_{\beta'})$ for all β, β', γ with $U_\gamma \subset U_\beta \cap U_{\beta'}$ there exists a unique $f_\alpha \in F(U_\alpha)$ such that $\rho_{\beta\alpha}(f_\alpha) = f_\beta$ for $\beta \in B$.

For a sheaf F on the base $\mathcal{U}$ and $p \in X$ define

$$F_p = \varinjlim_{U_\alpha \ni p} F(U_\alpha).$$

a) Show that the data

$$\mathcal{F}(U) = \left\{ (f_p \in F_p)_{p \in U} : \begin{array}{l} \text{for all } p \in U, \text{ there exists } U_\alpha \ni p,\ s \in F(U_\alpha) \\ \text{with } s_q = f_q \text{ for all } q \in U_\alpha \end{array} \right\}$$

defines a sheaf on X.

b) Show that the natural map $F(U_\alpha) \to \mathcal{F}(U_\alpha), f \mapsto (f_p)_{p \in U_\alpha}$ is an isomorphism for all $\alpha \in A$.

c) Prove that for any other sheaf $\mathcal{G}$ on X with isomorphisms $\mathcal{G}(U_\alpha) \cong F(U_\alpha)$ compatible with the restriction maps on both sides, we have $\mathcal{F} \cong \mathcal{G}$ (so $\mathcal{F}$ is the unique sheaf with this property, up to isomorphism). Conclude that for a ring R, we have $\tilde{R} = \mathcal{O}_{\operatorname{Spec}(R)}$.

Due June 03.