Algebraic Geometry

D-MATH, FS 2016

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Exercise Sheet 2

1. Prove that the radical of a homogeneous ideal in $\mathbb{C}[x_0, ..., x_n]$ is homogeneous.

2. Let $V \subset \mathbb{C}^n$ be an affine algebraic variety and let $A = \mathbb{C}[x_1, \ldots, x_n]/I(V)$ be the ring of regular functions on V. Let $g \in A$ and let $V_g \subset V$ be the open set defined by

$$V_g = \{p \in V \mid g(p) \neq 0\}$$

a) Show that, V_g is isomorphic to an affine algebraic variety.

b) What is the ring of regular functions on V_g? (Hint: It is the localization of A at g. Prove this)

3. Determine the ring of algebraic functions on $\mathbb{C}^2 \setminus \{0\}$. Show that it is a quasi-projective variety which is not isomorphic to an affine algebraic variety.

4. For $n, m \geq 1$, let $\mathbb{P}^{nm-1}$ be viewed as the projective space of $n \times m$-matrices. Prove that the locus of matrices of rank exactly k is a quasi-projective variety, denoted $R_k \subset \mathbb{P}^{nm-1}$.

5. Recall that $\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\}) / \sim$ with $v \sim \lambda v$ for $\lambda \in \mathbb{C}^*$. Thus every element of $\mathbb{P}^n$ can be represented as $[v]$ for $v \in \mathbb{C}^{n+1} \setminus \{0\}$.

a) Show that the group GL_{n+1} of invertible $(n+1) \times (n+1)$-matrices acts on $\mathbb{P}^n$ by $A[v] = [Av]$ for $A \in \mathrm{GL}_n$, $v \in \mathbb{C}^{n+1} \setminus \{0\}$.

b) A finite set $S \subset \mathbb{P}^n$ is called *in general linear position* if for all distinct $[v_1], \ldots, [v_k] \in S$ with $k \leq n+1$ we have that $v_1, \ldots, v_k$ are linearly independent. Show that for $p_1, \ldots, p_{n+2} \in \mathbb{P}^n$ in general linear position, there is $A \in \mathrm{GL}_{n+1}$ with

$$Ap_1 = [e_1], Ap_2 = [e_2], \ldots, Ap_{n+1} = [e_{n+1}], Ap_{n+2} = [e_1 + \ldots + e_{n+1}],$$

where $e_1, \ldots, e_{n+1}$ are the basis vectors of $\mathbb{C}^{n+1}$. Show that A is unique up to scaling by elements in $\mathbb{C}^*$.

6. Let $f \in \mathbb{C}[x_0, \ldots, x_n]$ be a homogeneous polynomial of degree $m > 0$. Then its vanishing set $V(f) \subset \mathbb{P}^n$ is called a *hypersurface of degree* m in projective space.

a) Show that the homogeneous polynomials $\mathbb{C}[x_0, \ldots, x_n]_m$ of degree m form a vector subspace of dimension $\binom{m+n}{n}$.

b) For any subset $S \subset \mathbb{P}^n$ with at most $\binom{m+n}{n} - 1$ elements, show that there is a hypersurface of degree m containing them.

c) In the case of conics ($m = 2$) in the projective plane ($n = 2$), show that through any 5 points in general linear position there is a unique hypersurface of degree 2.

Due March 11.