Algebraic Geometry

D-MATH, FS 2016

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Exercise Sheet 3

1. Prove the following basic facts about algebraic maps.

a) For $f : X \to Y$ and $g : Y \to Z$ algebraic morphisms of quasi-projective varieties, the composition $g \circ f : X \to Z$ is algebraic.

b) For affine varieties $V \subset \mathbb{C}^n$, $W \subset \mathbb{C}^m$, the algebraic maps $V \to W$ are exactly given by

$$f : V \to \mathbb{C}^m, v \mapsto (f_1(v), \ldots, f_m(v)),$$

where $f_1, \ldots, f_m \in \Gamma(V)$ are algebraic functions on V such that the image of the map f above is contained in W.

c) For quasi-projective varieties X, Y and a cover $X = \bigcup_{i \in I} V_i$ of X with open subsets V_i, a map $\varphi : X \to Y$ is algebraic if and only if the restriction $\varphi|_{V_i}$ is algebraic for all $i \in I$. In particular, for an open cover $Y = \bigcup_{i \in I} W_i$ it suffices to check that $\varphi^{-1}(W_i)$ is open in X and that

$$\varphi|_{\varphi^{-1}(W_i)} : \varphi^{-1}(W_i) \to W_i$$

is algebraic for all $i \in I$.

d) Show that the statement in b) remains true if V is a quasi-projective variety.

e) If $X \subset \mathbb{P}^n$ is a quasi-projective variety and if $F_0, \ldots, F_m \in \mathbb{C}[Z_0, \ldots, Z_m]$ are homogeneous polynomials of the same degree, which do not have a common zero on X, then

$$F : X \to \mathbb{P}^m, \xi \mapsto [F_0(\xi), \ldots, F_m(\xi)]$$

defines an algebraic morphism.

2. (a) Show that the algebraic isomorphisms $\mathbb{C} \to \mathbb{C}$ are exactly given by

$$x \mapsto ax + b \text{ for } a \in \mathbb{C}^*, b \in \mathbb{C}.$$

(b) Show that the algebraic isomorphisms $\mathbb{P}^1 \to \mathbb{P}^1$ are exactly given by

$$\varphi_A : [x, y] \mapsto [ax + by, cx + dy] \text{ for } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2.$$

3. Find with proof all integer solutions (x, y, z) of the equation $x^2 + y^2 = z^2$.

Due March 18.