Algebraic Geometry

**D-MATH, FS 2016**

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# Exercise Sheet 4

**1.** a) Develop a theory of algebraic subvarieties of $\mathbb{P}^n \times \mathbb{P}^m$ using bihomogeneous polynomials. For this you must define the Zariski topology and regular functions.

b) Prove that the projections $\mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^n$ and $\mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^m$ are algebraic maps.

c) The set $\mathbb{P}^n \times \mathbb{P}^m$ is covered by the sets $U_i \times U_j \cong \mathbb{C}^n \times \mathbb{C}^m = \mathbb{C}^{n+m}$ for $i = 0, \ldots, n$, $j = 0, \ldots, m$. Verify that a set $X \subset \mathbb{P}^n \times \mathbb{P}^m$ is Zariski-closed if and only if $X \cap U_i \times U_j \subset \mathbb{C}^{n+m}$ is Zariski-closed for all $i, j$. For such $X$, verify that a function $f : V \to \mathbb{C}$ from an open subset $V \subset X$ is algebraic if and only if its restriction to $V \cap U_i \times U_j$ is algebraic for all $i, j$.

**2.** For $n, m \geq 1$ integers, the *Segre-embedding* of $\mathbb{P}^n \times \mathbb{P}^m$ in $\mathbb{P}(\mathbb{C}^{n+1} \otimes \mathbb{C}^{m+1}) = \mathbb{P}^{(n+1)(m+1)-1}$ is the map

$$\sigma : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1}$$
$$([x_0, \ldots, x_n], [y_0, \ldots, y_m]) \mapsto \left[ (x_i y_j)_{\substack{i=0,\ldots,n \\ j=0,\ldots,m}} \right].$$

a) Show that $\sigma$ is a well-defined algebraic morphism — it is continuous in the Zariski topology and takes regular functions to regular functions.

b) Find equations for the image of the Segre embedding.

c) Prove that $\mathbb{P}^n \times \mathbb{P}^m$ is isomorphic to the image of the Segre embedding. In particular, show that the Segre embedding is bijective onto the image and the two have the same Zariski topology and the same notion of regular functions.

**3.** For $d, n \geq 1$, the *d-th Veronese embedding* of $\mathbb{P}^n$ is the map

$$\nu_d : \mathbb{P}^n \longrightarrow \mathbb{P}^{\binom{n+d}{n}-1}$$

whose homogeneous coordinates are given by the $\binom{n+d}{n}$ monomials of degree $d$ in the coordinates $x_0, \ldots, x_n$ of $\mathbb{P}^n$. For instance if $n = 2, d = 2$, we have

$$\nu_2 : \mathbb{P}^2 \to \mathbb{P}^5, [x_0, x_1, x_2] \mapsto [x_0^2, x_0 x_1, x_0 x_2, x_1^2, x_1 x_2, x_2^2].$$

Prove that $\nu_d$ is an algebraic map. Show that the image of $\nu_d$ is defined by quadratic equations (find the equations).

**4.** A *rational normal curve* in $\mathbb{P}^n$ is the composition of $\nu_n : \mathbb{P}^1 \to \mathbb{P}^n$ with a projectively linear transformation, i.e. a map of the form $p \mapsto Ap$ for $A \in \mathrm{GL}_{n+1}$. Let $p_0, ..., p_{n+2}$ be $n+3$ points in general linear position in $\mathbb{P}^n$. Prove that there exists a rational normal curve which passes through the points $p_i$.

**5*.** Show that for $V \subset \mathbb{C}^m$, $W \subset \mathbb{C}^n$ affine varieties, the set $V \times W \subset \mathbb{C}^{m+n}$ is an affine variety. Compute $I(V \times W)$ and show that the coordinate ring $\Gamma(V \times W)$ is given by $\Gamma(V) \otimes_{\mathbb{C}} \Gamma(W)$.

**Due April 4.**