Algebraic Geometry

D-MATH, FS 2016

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Exercise Sheet 6

1. Let $n \geq 2$ and let $\omega \in \bigwedge^2 \mathbb{C}^n$ be nonzero.

a) Show that there exists $k \geq 1$ such that

$$\omega = v_1 \wedge v_2 + \ldots + v_{2k-1} \wedge v_{2k}$$

for $v_1, \ldots, v_{2k} \in \mathbb{C}^n$ linearly independent.

b) Prove that

$$\omega^{\wedge k} = \underbrace{\omega \wedge \omega \wedge \ldots \wedge \omega}_{k \text{ times}} \neq 0$$

but $\omega^{\wedge k+1} = 0$. In particular, k is uniquely determined and it is called the *rank* of ω. If ω is of rank 1, it is called *decomposable*.

c) Consider the map

$$s(\omega) : \mathbb{C}^n \to \bigwedge^3 \mathbb{C}^n, v \mapsto v \wedge \omega.$$

Show that it has trivial kernel if the rank of ω is greater than 1 and that the kernel has dimension 2 if the rank is equal to 1.

d) Let $\mathrm{Gr}(2, n)$ be the Grassmannian of 2-planes in $\mathbb{C}^n$ and let

$$\gamma : \mathrm{Gr}(2, n) \to \mathbb{P}(\bigwedge^2 \mathbb{C}^n)$$

be the Plücker embedding given by $\gamma(\Lambda) = [v_1 \wedge v_2]$, where $\Lambda \in \mathrm{Gr}(2, n)$ and $v_1, v_2 \in \mathbb{C}^n$ is a basis of Λ.

Show that γ is a bijection onto the set X of classes of decomposable vectors in $\mathbb{P}(\bigwedge^2 \mathbb{C}^n)$.

e) Let $(e_i)_{i=1,\ldots,n}$ be the standard basis on $\mathbb{C}^n$. Let $(e_i \wedge e_j)_{1 \leq i < j \leq n}$ be the induced basis for $\bigwedge^2 \mathbb{C}^n$. Writing each element $\omega \in \bigwedge^2 \mathbb{C}^n$ as $\omega = \sum_{i<j} a_{ij} e_i \wedge e_j$ induces homogeneous coordinates $[(a_{ij})_{i<j}]$ on $\mathbb{P}(\bigwedge^2 \mathbb{C}^n)$. Use *b)* to show that X is cut out by quadratic equations in the coefficients a_{ij}.

In particular, for $n = 4$ we have that X is cut out by the famous Plücker quadric

$$X = V(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}).$$

2. Let $L \subset \mathbb{P}^3$ be a fixed line. Let $H_L \subset \mathrm{Gr}(2, 4)$ be the locus of lines which meet L.

a) Consider the incidence subset

$$I = \{(p,l) \in \mathbb{P}^3 \times \mathrm{Gr}(2,4) : p \in l\} \subset \mathbb{P}^3 \times \mathrm{Gr}(2,4).$$

Show that I is a closed subvariety of $\mathbb{P}^3 \times \mathrm{Gr}(2,4)$.

b) Conclude that that H_L is a closed subvariety of $\mathrm{Gr}(2,4)$.

c) What is the image of H_L under the Plücker embedding in $\mathbb{P}^5$?

d) Use this geometry to give a different proof of the fact that there are 2 lines meeting 4 fixed general lines in space.

3. Let $Q_2 \subset \mathbb{P}^3$ be the quadric given by the equation $x_0x_3 - x_1x_2$. Find (with proof) all lines which lie on Q_2.

4. Let $X \subset \mathbb{P}(\bigwedge^2 \mathbb{C}^4) = \mathbb{P}^5$ be the quadric which is the image of the Plücker embedding of $\mathrm{Gr}(2,4)$. Find all planes $\mathbb{P}^2 \subset \mathbb{P}^5$ (linearly embedded) which lie on X. Can you interpret these in terms of the geometry of $\mathrm{Gr}(2,4)$?

5. a) Let $X, Y \subset \mathbb{C}^n$ be subvarieties and assume Y is cut out by polynomials $f_1, \ldots, f_r \in \mathbb{C}[x_1, \ldots, x_n]$. Then for $p \in X \cap Y$ show that $\dim T_pX \cap Y \geq \dim T_pX - r$.

b) Consider $X = V(xy - z^2) \subset \mathbb{C}^3$. Find irreducible curves $C_1, C_2, C_3 \subset X$ going through the origin and satisfying $\dim T_0C_i = i$ for $i = 1, 2, 3$.

c) Show that the line $l \subset X$ spanned by $(0,1,0)$ is not cut out from X by a single polynomial.

Due April 22.