Algebraic Geometry

D-MATH, FS 2016

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Exercise Sheet 7

1. Prove that the Zariski tangent space at the point $[S] \in \mathrm{Gr}(r, V)$ is canonically isomorphic to $S^* \otimes V/S$ (or equivalently to $\mathrm{Hom}(S, V/S)$).

2. Let $M_{n,n}$ be the space of $n \times n$ complex matrices. Let $\det : M_{n,n} \to \mathbb{C}$ be the determinant and let $Z = V(\det)$ be its zero set. Find all the points of Z such that the Zariski tangent space to Z has dimension exactly $n^2 - 1$.

3. In this exercise, we want to show the following result about intersections of subvarieties of $\mathbb{P}^n$.

Lemma 1. *Let $X, Y \subset \mathbb{P}^n$ be subvarieties such that* $\dim(X) + \dim(Y) \geq n$. *Then $X \cap Y \neq \emptyset$.*

Below, we will first prove this result in the case that Y is a linear subspace of $\mathbb{P}^n$ and then see how to show the general case from this.

a) For $X \subset \mathbb{P}^n$ a subvariety let

$$\hat{X} = \{p \in \mathbb{C}^{n+1} \setminus \{0\} : [p] \in X\} \cup \{0\} \subset \mathbb{C}^{n+1}$$

be the *affine cone* over X. Show that $\hat{X}$ is an affine variety and that it is irreducible if X is irreducible. *Hint:* For irreducibility, given open sets V_1, V_2 in $\hat{X}$, construct open sets in X by intersecting with an affine hyperplane in $\mathbb{C}^{n+1}$.

b) Prove that $\dim(\hat{X}) = \dim(X) + 1$. *Hint:* Look at the intersections of $\hat{X}$ with $\hat{U}_i = \{(x_0, \ldots, x_n) : x_i \neq 0\}$.

c) Let $L \subset \mathbb{P}^n$ be a linear subspace of codimension c (i.e. dimension $n - c$). Show that $\dim(X \cap L) \geq \dim(X) - c$. In particular, if $\dim(X) \geq c$ we have $X \cap L \neq \emptyset$. You may use the following result, a geometric version of Krull's principal ideal theorem.

Theorem 1. *If X is an affine variety and f is a regular function on X, then* $\dim(V(f)) \geq \dim(X) - 1$.

d) Let $X, Y \subset \mathbb{P}^n$ be subvarieties. Consider $\mathbb{P}^{2n+1}$ with coordinates $x_0, \ldots, x_n$, $y_0, \ldots, y_n$ and identify $X \subset \mathbb{P}^n = V(y_0, \ldots, y_n) \subset \mathbb{P}^{2n+1}$ and similarly $Y \subset \mathbb{P}^n = V(x_0, \ldots, x_n) \subset \mathbb{P}^{2n+1}$. Consider the *join* $J(X, Y)$ of X and Y defined by

$$J(X, Y) = \{[t(x, 0) + s(0, y)] : [x] \in X, [y] \in Y, (s, t) \in \mathbb{C} \setminus \{0\}\} \subset \mathbb{P}^{2n+1}.$$

Show that $J(X, Y)$ is a subvariety of dimension $\dim(X) + \dim(Y) + 1$. Here you can use the following result, which will be proved later.

Theorem 2. *If X, Y are irreducible quasi-projective varieties, we have* $\dim(X \times Y) = \dim(X) + \dim(Y)$.

e) Identify $X \cap Y$ as an intersection of $J(X, Y)$ with $n + 1$ hyperplanes in $\mathbb{P}^{2n+1}$. Conclude that it is nonempty if $\dim(X) + \dim(Y) \geq n$.

f) Give an example of a projective variety Z with closed subvarieties X, Y satisfying $\dim(X) + \dim(Y) \geq \dim(Z)$ but $X \cap Y = \emptyset$.

4. Prove that for any five lines $L_1, \ldots, L_5 \subset \mathbb{P}^2$ in the projective plane there is a conic C tangent to all of them. Here we mean that for all i there is a point $p \in C \cap L_i$ such that $T_pL_i \subset T_pC$. *Hint:* Identify the space of plane conics with $\mathbb{P}^5$ and show that the set of conics tangent to a fixed line is a quadric hypersurface in $\mathbb{P}^5$.

Due April 29.