Exercise Sheet 8

1. Consider the category of

(*) affine algebraic varieties with algebraic maps,

(**) finitely generated reduced $\mathbb{C}$-algebras with $\mathbb{C}$-algebra homomorphism.

Prove the equivalence of the categories (*) and (**) via the contravariant functor

$$X \to \Gamma(X)$$

2. Let X be an affine variety with coordinate ring A.

a) Show that X is irreducible iff A is a domain.

b) Recall that by the Nullstellensatz, the operations $Y \mapsto I(Y)$ and $I \mapsto V(I)$ define inverse bijections

$$\{\text{closed subsets } Y \subset X\} \leftrightarrow \{\text{radical ideals } I \subset A\}.$$

Show that this bijection restricts to bijections

$$\{\text{irreducible closed subsets } Y \subset X\} \leftrightarrow \{\text{prime ideals } I \subset A\},$$
$$\{\text{points } p \in X\} \leftrightarrow \{\text{maximal ideals } m \subset A\}.$$

3. Let X be a quasi-projective variety and $p \in X$. Define

$$\mathcal{O}_{X,p} = \{(f, U) : p \in U \subset X \text{ nonempty open}, \ f : U \to \mathbb{C} \text{ algebraic}\}/\sim,$$

where $(f, U) \sim (g, V)$ if there exists a nonempty open neighborhood $W \subset U \cap V$ of p with $f|_W = g|_W$. Show that $\mathcal{O}_{X,p}$ has a natural $\mathbb{C}$-algebra structure. It is called the *ring of germs of algebraic functions* on X around p. If $V \subset X$ is an open neighborhood of p, show that there is a natural isomorphism $\mathcal{O}_{X,p} \cong \mathcal{O}_{V,p}$. If V is affine with coordinate ring A and if $p \in V$ corresponds to the maximal ideal $m \subset A$, show that $\mathcal{O}_{V,p}$ is isomorphic to the localization A_m.

4. Let X be an affine algebraic variety and let A be the ring of algebraic functions on X. Let $p \in X$ be a point and let $\mathfrak{m} \subset A$ be the associated maximal ideal. Let $A_{\mathfrak{m}}$ be the localization of A at $\mathfrak{m}$. Let $\mathfrak{m}A_{\mathfrak{m}}$ be the maximal ideal of $A_{\mathfrak{m}}$. Prove that the natural map

$$\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}A_{\mathfrak{m}}/\mathfrak{m}^2 A_{\mathfrak{m}}$$

is an isomorphism of $A/\mathfrak{m}$ vector spaces.

5. Let X and Y be irreducible quasi-projective varieties. Recall that X and Y are birational if there are nonempty open sets $U \subset X$, $W \subset Y$, such that U is isomorphic to W. Prove that X and Y are birational if and only if $K(X)$ is isomorphic to $K(Y)$.

Due May 06.