Exercise Sheet 9

1. Let V_d be the $\mathbb{C}$-vector space of all degree d homogeneous polynomials in $\mathbb{C}[x,y]$. Let $L_1, L_2, \ldots$ and $M_1, M_2, \ldots$ be sequences of nonzero linear forms (of the type $ax + by$ for $a, b \in \mathbb{C}$), and assume no L_i is a scalar multiple of some M_j, i.e. $L_i = \lambda M_j$ for $\lambda \in \mathbb{C}^*$. Show that

$$\{A_{i,j} := L_1 L_2 \ldots L_i M_1 M_2 \ldots M_j | i + j = d\}$$

form a $\mathbb{C}$-basis for V_d. *Hint:* Use induction on d.

2. Let $\mathfrak{m}$ be the ideal of $\mathbb{C}[x,y]$ generated by x and y, and let $\mathbb{C}[x,y]_{\mathfrak{m}}$ be the corresponding local ring. Assume polynomials $F(x,y)$ and $G(x,y)$ vanishing at $(0,0)$ have no common factor $H(x,y) \in \mathbb{C}[x,y]$ with $H(0,0) = 0$.

a) Use Hilbert's Nullstellensatz to show that there exists a positive integer N such that
$$x^N, y^N \in (F,G) \cdot \mathbb{C}[x,y]_{\mathfrak{m}}.$$

b) Show that there exists a positive integer M such that
$$\mathfrak{m}^M \subset (F,G) \cdot \mathbb{C}[x,y]_{\mathfrak{m}}.$$

3. The purpose of this exercise is to complete the proof of the existence of $I_P(F,G)$ in the class.

Assume the point P is $(0,0) \in \mathbb{C}^2$. A polynomial $F \in \mathbb{C}[x,y]$ which vanishes at $(0,0)$ can be uniquely expressed as

$$F = F_p + \tilde{F}$$

with F_p ($p > 0$) a homogeneous polynomial of degree p, such that each term of $\tilde{F}$ has degree $> p$. Similarly we express $G \in \mathbb{C}[x,y]$ which vanishes at $(0,0)$ as

$$G = G_q + \tilde{G}$$

with G_q ($q > 0$) a homogeneous polynomial of degree q. Let $\mathfrak{m} = (x,y)$ be the maximal ideal in $\mathbb{C}[x,y]$. We define

$$I(F,G) := \dim_{\mathbb{C}}\Big(\mathbb{C}[x,y]_{\mathfrak{m}}/(F,G)\Big).$$

a) Show that
$$I(F,G) \geq \dim_{\mathbb{C}}\Big(\mathbb{C}[x,y]/(F,G,\mathfrak{m}^{p+q})\Big). \tag{1}$$

b) Prove that the above inequality is an equality if and only if

$$\mathfrak{m}^{p+q} \cdot \mathbb{C}[x,y]_{\mathfrak{m}} \subset (F,G) \cdot \mathbb{C}[x,y]_{\mathfrak{m}}. \tag{2}$$

Recall that for F, G as before we have introduced the following exact sequence

$$\mathbb{C}[x,y]/\mathfrak{m}^p \times \mathbb{C}[x,y]/\mathfrak{m}^q \xrightarrow{\varphi} \mathbb{C}[x,y]/\mathfrak{m}^{p+q} \xrightarrow{\pi} \mathbb{C}[x,y]/(F,G,\mathfrak{m}^{p+q}) \to 0.$$

Here the first map φ is given by $\varphi(\overline{f}, \overline{g}) = \overline{fG + gF}$ for $f, g \in \mathbb{C}[x,y]$[1], and the second map π is the natural projection.

c) Use the exact sequence above to show that

$$\dim_{\mathbb{C}}\Big(\mathbb{C}[x,y]/(F,G,\mathfrak{m}^{p+q})\Big) \geq pq. \tag{3}$$

Moreover, show that the equality occurs if and only if φ is injective.

Combining (1) and (3), We have proved that

$$I(F,G) \geq pq. \tag{4}$$

d) Prove directly that φ is injective if and only if F_p and G_q have no common factor. (The latter is equivalent to the condition that F and G have no common tangent line at $(0,0)$.)

Now assume F_p and G_q have **NO** common factor. Let $F_p = L_1L_2\ldots L_p$ and $G_q = M_1M_2\ldots M_q$ with L_i, M_j linear forms, and let $L_i = L_p$ for $i > p$ and $M_j = M_q$ for $j > q$. Define

$$A_{i,j} := L_1L_2\ldots L_iM_1M_2\ldots M_j \in \mathbb{C}[x,y]$$

as in Exercise 1.

e) For an integer $t > 0$, prove that

$$\mathfrak{m}^t \cdot \mathbb{C}[x,y]_{\mathfrak{m}} \subset (F,G) \cdot \mathbb{C}[x,y]_{\mathfrak{m}}$$

if and only if $A_{i,j} \in (F,G) \cdot \mathbb{C}[x,y]_{\mathfrak{m}}$ when $i + j \geq t$. *Hint:* Use Exercise 1.

f) Assume $t \geq p + q$, and

$$\mathfrak{m}^{t+1} \cdot \mathbb{C}[x,y]_{\mathfrak{m}} \subset (F,G) \cdot \mathbb{C}[x,y]_{\mathfrak{m}}.$$

Then prove $A_{i,j} \in (F,G) \cdot \mathbb{C}[x,y]_{\mathfrak{m}}$ when $i + j \geq t$. *Hint:* The inequality $i + j \geq p + q$ implies that $i \geq p$ or $j \geq q$. If $i \geq p$, we have $A_{i,j} = (F - \tilde{F})\tilde{A}$ with $\tilde{A} \in \mathbb{C}[x,y]$.

g) Prove that (4) is an equality if and only if F_p and G_q have no common factor. *Hint:* This is a direct consequence of the results above. You may need to use Exercise 2.

Due May 13.

[1]For $F \in \mathbb{C}[x,y]$, we always use $\overline{F}$ to denote its residue class in $\mathbb{C}[x,y]/I$ for some ideal I.