Algebraic Geometry

D-MATH, FS 2016

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Exercise Sheet 0 - Solutions

1. Sketch the following subsets of $\mathbb{R}^2$ defined by polynomial equations in the coordinates x, y.

(a) $V(x^2 - x)$

(b) $V(x^2 + 4y^2 - 4)$

(c) $V(x^2 + 4y^2 + 4)$

(d) $V(xy - 1)$

(e) $V(xy)$

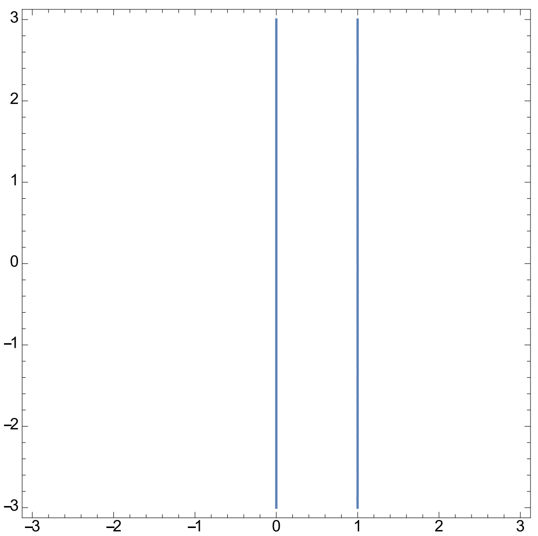
(f) $V(y^2 - x^3)$

(g) $V(y^2 - x^2(x + 1))$

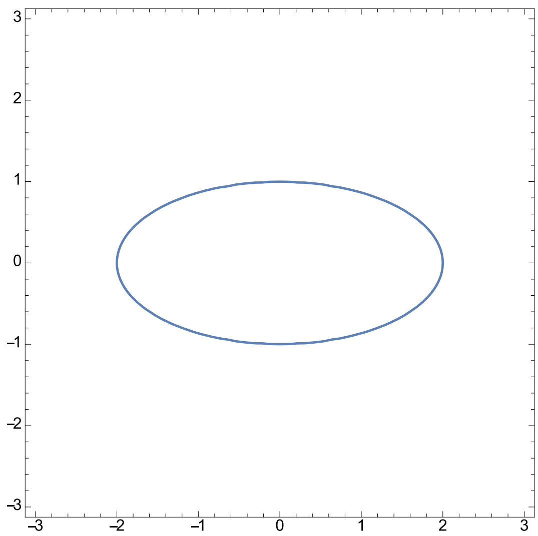
(h) $V(y^2 - x^2(x - 1))$

(i) $V(xy - 1, 2y + 2x - 5)$

Solution (a) $V(x^2 - x) = \{-1, 0\} \times \mathbb{R}$ is the union of two vertical lines.

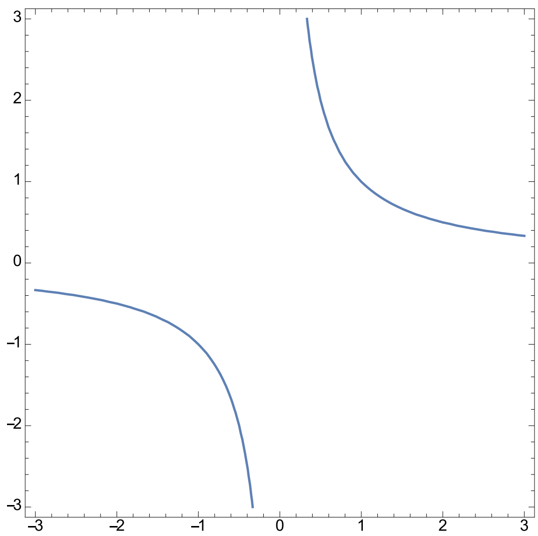


(b) $V(x^2 + 4y^2 - 4)$ is an ellipse with half axes of length 2 in x-direction and length 1 in y-direction.

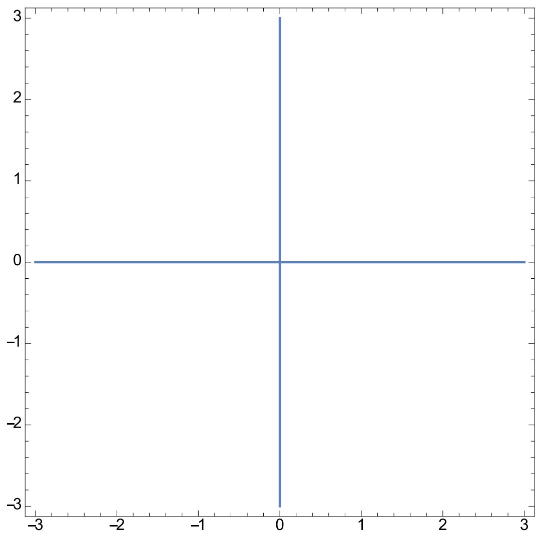


(c) $V(x^2 + 4y^2 + 4) = \emptyset$, as $x^2 + y^2 + 4 \geq 4 > 0$ for all $x, y \in \mathbb{R}$.

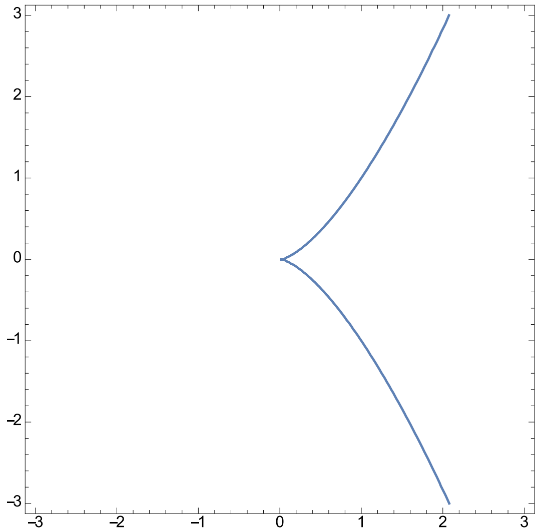
(d) $V(xy - 1)$ is the standard **hyperbola**.



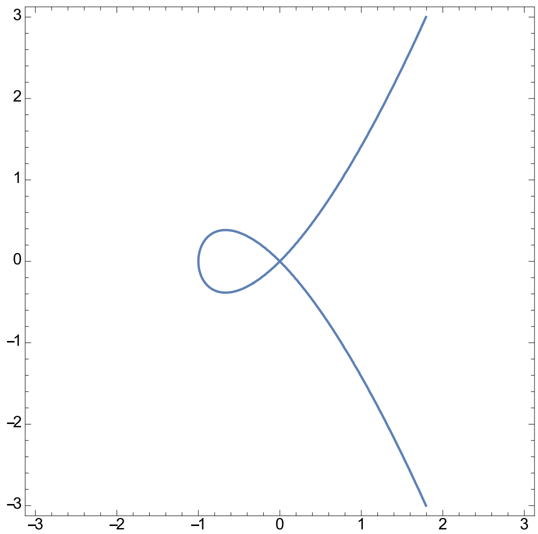
(e) $V(xy)$ is the union of the x- and y-axis.



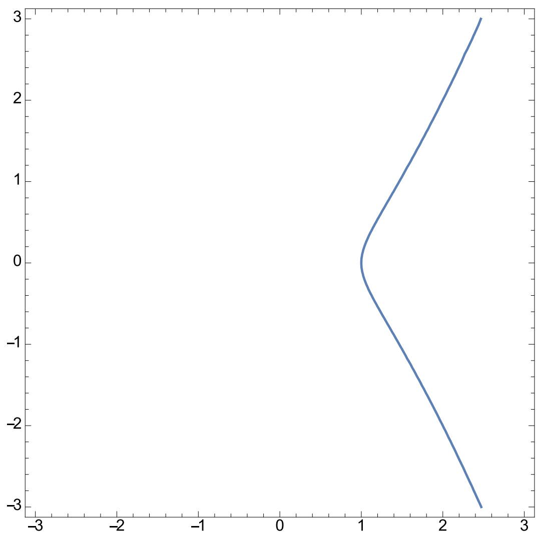
(f) $V(y^2 - x^3)$ is a curve with a **cusp** at $(0, 0)$.



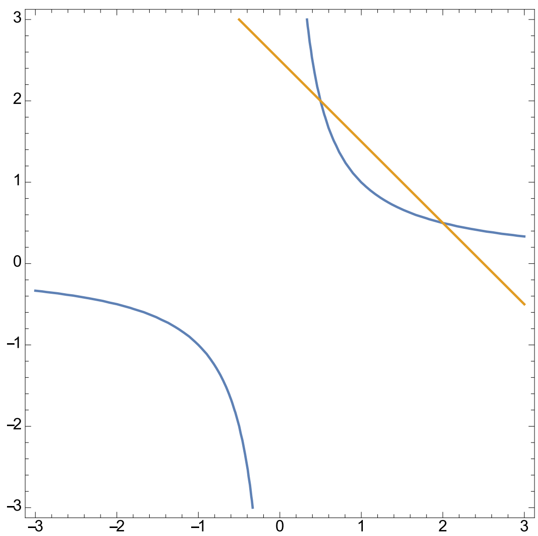
(g) $V(y^2 - x^2(x + 1))$ is a curve with a **node** at $(0, 0)$.



(h) $V(y^2 - x^2(x - 1))$ has two connected components: one is a curve intersecting the x-axis at $x = 1$, the other is an isolated point at the origin (the point cannot be seen in the picture below).



(i) $V(xy-1, 2y+2x-5)$ is the intersection of the hyperbola from above with the line $y = -x + 5/2$, so it consists of the points $(1/2, 2), (2, 1/2)$.



2. Let $n \geq 1$ and $I, J, I_\alpha \subset \mathbb{C}[x_1, \ldots, x_n]$ be ideals ($\alpha \in A$).

(a) Show that $V(I \cdot J) = V(I \cap J) = V(I) \cup V(J)$, where $I \cdot J = (rs : r \in I, s \in J)$ is the ideal generated by products of elements of I and J.

(b) Show that $V(\sum_{\alpha \in A} I_\alpha) = \bigcap_{\alpha \in A} V(I_\alpha)$.

(c) Conclude, that the sets $V(I) \subset \mathbb{C}^n$ (for $I \subset \mathbb{C}[x_1, \ldots, x_n]$) form the closed sets of a topology.

(d) Give an example of n and a family I_α as above, such that

$$V(\bigcap_{\alpha \in A} I_\alpha) \neq \bigcup_{\alpha \in A} V(I_\alpha).$$

Solution (a) Let $p \in V(I) \cup V(J)$ and assume, without loss of generality, $p \in V(I)$. Then every function $r \cdot s$ for $r \in I, s \in J$ vanishes on p, as already $r(p) = 0$ and of course also all functions in $I \cap J$ vanish on p (as $I \cap J \subset I$).
On the other hand, assume $p \notin V(I) \cup V(J)$, then we find $r \in I, s \in J$

with $r(p) \neq 0, s(p) \neq 0$. But the product rs is in IJ by definition, and also in $I \cap J$, because I, J are ideals. Hence $p \notin V(IJ)$ and also $p \notin V(I \cap J)$.

(b) If $p \in \bigcap_{\alpha \in A} V(I_\alpha)$ then for all $\alpha \in A$ and $f \in I_\alpha$ we have $f(p) = 0$. As the ideal $\sum_{\alpha \in A} I_\alpha$ is generated by such f, all elements of this ideal vanish on p.
On the other hand, we have that for all $\alpha_0 \in A$ that $I_{\alpha_0} \subset \sum_{\alpha \in A} I_\alpha$, so if all functions $f \in \sum_{\alpha \in A} I_\alpha$ vanish on p, then necessarily all functions in I_{α_0} vanish there, so $p \in V(I_{\alpha_0})$ for all $\alpha_0 \in A$. This shows the other inclusion.

(c) What we showed above is that finite unions and arbitrary intersections of closed sets of the proposed topology remain in the topology. Using that $V(\{0\}) = \mathbb{C}^n$ and $V(\mathbb{C}[x_1, \ldots, x_n]) = \emptyset$ (as $V(1) = \emptyset$) we have verified the axioms for (the closed subsets in) a topology.

(d) We take $n = 1$ and $A = \mathbb{Z}$ with the ideal $I_m = (x - m)$. Then $V(I_m) = \{m\}$ and $\bigcup_{m \in \mathbb{Z}} V(I_m) = \mathbb{Z} \subset \mathbb{C}$. Now any polynomial $f \in \mathbb{C}[x]$ contained in all the ideals I_m must vanish at all the points $m \in \mathbb{Z}$. But as a nonzero polynomial of degree d can have at most d zeroes, it follows that $f = 0$. Hence

$$V(\bigcap_{m \in \mathbb{Z}} I_m) = V(\{0\}) = \mathbb{C} \neq \bigcup_{m \in \mathbb{Z}} V(I_m) = \mathbb{Z}.$$

To be discussed in the exercise class.