Algebraic Geometry
D-MATH, FS 2016

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Exercise Sheet 1 - Solutions

1. Give a careful proof that the ring of functions on an affine, algebraic variety $V \subset \mathbb{C}^n$ is given by $\mathbb{C}[x_1, \ldots, x_n]/I(V)$.

Solution

Let $\Gamma(V)$ be the ring of algebraic functions on V. Of course every function given by a polynomial in $\mathbb{C}[x_1, \ldots, x_n]$ restricts to an algebraic function on V. The kernel of this restriction map

$$\mathbb{C}[x_1, \ldots, x_n] \to \Gamma(V)$$

is exactly the ideal $I = I(V)$, so we have a natural inclusion $\mathbb{C}[x_1, \ldots, x_n]/I \subset \Gamma(V)$. It remains to show that every regular function $f \in \Gamma(V)$ is the restriction of a polynomial.
By definition, for every $p \in V$ we find an open subset $W_p \subset V$ containing p and $g_p, h_p \in \mathbb{C}[x_1, \ldots, x_n]$ such that g_p does not vanish on W_p and $f|_{W_p} = h_p/g_p$. As the sets $U_r = \{q \in V : r(q) \neq 0\}$ (for $r \in \mathbb{C}[x_1, \ldots, x_n]$) form a basis of the Zariski topology of V, we find a polynomial r_p with $p \in U_{r_p} \subset W_p$. Write $\tilde{h}_p = h_p r_p$ and $\tilde{g}_p = g_p r_p$ and note

$$U_{\tilde{g}_p} = U_{r_p g_p} = U_{r_p} \cap U_{g_p} = U_{r_p}$$

as by assumption $U_{g_p} \supset W_p \supset U_{r_p}$. We conclude

$$f|_{U_{\tilde{g}_p}} = \frac{h_p}{g_p} = \frac{h_p r_p}{g_p r_p} = \frac{\tilde{h}_p}{\tilde{g}_p}.$$

Now as $p \in U_{\tilde{g}_p}$, we have a covering

$$V = \bigcup_{p \in V} U_{\tilde{g}_p}.$$

This is equivalent to saying, that there is no point $q \in \mathbb{C}^n$ such that all functions of I vanish at q (so $q \in V$) and all functions $\tilde{g}_p$ vanish there. Hence

$$\emptyset = V(I + \sum_{p \in V} \tilde{g}_p).$$

By the Nullstellensatz, we have that

$$\mathbb{C}[x_1, \ldots, x_n] = I(\emptyset) = I(V(I + \sum_{p \in V} \tilde{g}_p)) = \sqrt{I + \sum_{p \in V} \tilde{g}_p}.$$

Thus some power of $1 \in \mathbb{C}[x_1, \ldots, x_n]$ is contained in $I + \sum_{p \in V} \tilde{g}_p$. But this simply means $1 \in I + \sum_{p \in V} \tilde{g}_p$. Hence, we find finitely many $p_1, \ldots, p_m$ such that

$$1 = t + s_1 \tilde{g}_{p_1} + \ldots + s_m \tilde{g}_{p_m},$$

where $t \in I$ and $s_1, \ldots, s_m \in \mathbb{C}[x_1, \ldots, x_n]$. Restricted to V we know that t vanishes by definition, so we have

$$1 = s_1 \tilde{g}_{p_1} + \ldots + s_m \tilde{g}_{p_m} \text{ on } V. \tag{1}$$

Automatically we have that the sets $U_{\tilde{g}_{p_i}}$ (for $i = 1, \ldots, m$) cover V. If there was a point $q \in V$, where they all vanish, then the entire right side of (1) vanishes at q, a contradiction.

To ease the notation, from now on we write $h_i = \tilde{h}_{p_i}$ and $g_i = \tilde{g}_{p_i}$. To relate the functions h_i, g_i for different i, we use that on the overlap $U_{g_i} \cap U_{g_j} = U_{g_i g_j}$ we have

$$\frac{h_i}{g_i}|_{U_{g_i g_j}} = f|_{U_{g_i g_j}} = \frac{h_j}{g_j}|_{U_{g_i g_j}}.$$

Thus $h_i g_j - h_j g_i$ vanishes on $U_{g_i g_j}$, so after multiplying with $g_i g_j$ it vanishes on all of V. We have

$$I \ni (h_i g_j - h_j g_i) g_i g_j = (h_i g_i)(g_j^2) - (h_j g_j)(g_i^2)$$

Now replace everywhere h_i by $h_i g_i$ and g_i by g_i^2, then we have $U_{g_i} = U_{g_i^2}$ and $f|_{U_{g_i}} = h_i / g_i = (h_i g_i)/(g_i^2)$. Thus after this replacement, we have $h_i g_j - h_j g_i \in I$ for all i, j. In other words

$$h_i g_j = h_j g_i \text{ on } V. \tag{2}$$

We now claim that the polynomial

$$P = \sum_{i=1}^{m} s_i h_i$$

agrees with f on all sets U_{g_j}, which shows that f is indeed the restriction of the polynomial P to V. But on U_{g_j} we have

$$P \cdot g_j|_{U_{g_j}} = \sum_{i=1}^{m} s_i h_i g_j \overset{(2)}{=} \sum_{i=1}^{m} s_i h_j g_i = \left(\sum_{i=1}^{m} s_i g_i \right) h_j \overset{(1)}{=} 1 \cdot h_j = h_j.$$

This shows $P|_{U_{g_j}} = h_j / g_j = f|_{U_{g_j}}$, finishing the proof.

2. For each of the following sets X, find the ideal $I(X)$ of functions vanishing on X and conclude that X is algebraic. Compute their rings of regular functions and show that they are pairwise non-isomorphic as $\mathbb{C}$-algebras.

(a) $X = \{p\} \subset \mathbb{C}^n$ for $p \in \mathbb{C}^n$

(b) $X = \{(t, t^2, t^3) : t \in \mathbb{C}\} \subset \mathbb{C}^3$ (the twisted cubic curve)

(c) $X = \{(t^2, t^3) : t \in \mathbb{C}\} \subset \mathbb{C}^2$ (a cuspidal curve)

Solution

(a) Let $p = (a_1, \dots, a_n) \in \mathbb{C}^n$ then we claim

$$I = (x_1 - a_1, \dots, x_n - a_n)$$

is the ideal of functions vanishing on p. Indeed, it is clear that all functions $x_i - a_i$ vanish at p, so $I \subset I(\{p\})$. Also it is clear that $V(I) = \{p\}$. Now assume we have $f \in \mathbb{C}[x_1, \dots, x_n]$ which vanishes at p. We will show $f \in I$ by proving that $f \equiv 0$ modulo the ideal I. But note that by definition $x_i \equiv a_i$ modulo I. Hence, we may replace every occurrence of the variable x_i in f by the complex number a_i, so

$$f(x_1, \dots, x_n) \equiv f(a_1, \dots, a_n) = f(p) = 0 \bmod I.$$

Thus $I(\{p\}) = I$. For any $f \in \mathbb{C}[x_1, \dots, x_n]$ the argument above also shows that $f \equiv f(p)$ modulo I. This immediately implies that the ring of algebraic functions on $\{p\}$ is

$$\mathbb{C}[x_1, \dots, x_n]/(x_1 - a_1, \dots, x_n - a_n) = \mathbb{C},$$

as expected (a function on a point simply assigns a complex value to this point).

(b) Let w, x, y be coordinates on $\mathbb{C}^3$, then the ideal of the twisted cubic curve is given by

$$I = (w^2 - x, w^3 - y).$$

Indeed, the functions $w^2 - x, w^3 - y$ vanish on X and even $V(I) = X$. Conversely, we have an isomorphism

$$\begin{aligned} \mathbb{C}[w, x, y]/(w^2 - x, w^3 - y) &\xrightarrow{\sim} \mathbb{C}[w] \\ f(w, x, y) &\mapsto f(w, w^2, w^3) \end{aligned}$$

by the same argument as above (we replace x by w^2 and y by w^3 everywhere). If $f(w, x, y) \in \mathbb{C}[w, x, y]$ vanishes on X, it maps to $f(w, w^2, w^3) = 0$ under the above isomorphism, so $f \equiv 0$ modulo I, hence $f \in I$. This shows $I(X) = (w^2 - x, w^3 - y)$.

(c) Let x, y be coordinates on $\mathbb{C}^2$, then the ideal of the cuspidal curve is given by

$$I = (y^2 - x^3).$$

Indeed, $y^2 - x^3$ vanishes on X and we claim that $V(y^2 - x^3) = X$. Indeed, let $x, y \in \mathbb{C}$ with $y^2 = x^3$ and chose $t \in \mathbb{C}$ with $t^2 = x$. Then $y^2 = (t^3)^2$, so $y = t^3$ or $y = -t^3 = (-t)^3$. Thus $(x, y) = (t^2, t^3)$ or $(x, y) = ((-t)^2, (-t)^3)$. On the other hand let $f(x, y)$ be a polynomial, whose class modulo I we want to compute. We use $y^2 \equiv x^3$ to replace all occurrences of powers $y^2, y^3, \dots$ by powers of x. We see, that as a complex vector space we have

$$\mathbb{C}[x, y]/I \cong \mathbb{C}[x] \oplus \mathbb{C}[x] \cdot y.$$

But any polynomial $f(x, y) = f_1(x) + f_2(x)y$ vanishing on X satisfies

$$f_1(t^2) + f_2(t^2)t^3 = 0.$$

Now note that in the polynomial $f_1(t^2)$ we only have even powers of t and in the polynomial $f_2(t^2)t^3$ only odd powers. Hence both f_1, f_2 must be zero for the above equation to be possible. This shows that $I(X) = I$.

We conclude by showing that the three rings above are all distinct. Indeed, the first ring $\mathbb{C}$ is finite dimensional over $\mathbb{C}$, in contrast to the other two, so it cannot coincide with one of them. Now $\mathbb{C}[w]$ is a unique factorization domain. But in $\mathbb{C}[x,y]/(y^2-x^3)$ we have $y \cdot y = x \cdot x^2$ but neither $y \equiv \lambda x$ nor $y \equiv \lambda x^2$ modulo $y^2 - x^3$ (for some $\lambda \in \mathbb{C}^*$) by a degree-argument. Indeed, all nonzero elements of the ideal (y^2-x^3) have degree at least 2 in y, but $y - \lambda x, y - \lambda x^2$ have degree 1 in y.

3. Show that the ring of algebraic functions on $\mathbb{CP}^n$ are the constant functions $\mathbb{C}$.

Solution

Cover $\mathbb{CP}^n$ with the standard covering $U_i = \{x_i \neq 0\}$ and recall that U_i is isomorphic to $\mathbb{C}^n$ with the coordinates $\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}$. Hence given an algebraic function f on $\mathbb{CP}^n$, we know that $f_{|U_i}$ is a polynomial f_i in $\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}$ (e.g. by problem 1).
Now consider two open subsets, lets say, U_0 and U_1. On the intersection we must have

$$f_0 = \sum_I a_I \left(\frac{x_1}{x_0}\right)^{i_1} \cdots \left(\frac{x_n}{x_0}\right)^{i_n} = \sum_J b_J \left(\frac{x_0}{x_1}\right)^{j_1} \left(\frac{x_2}{x_1}\right)^{j_2} \cdots \left(\frac{x_n}{x_1}\right)^{i_n} = f_1$$

The left hand side has only terms with nonnegative x_1 exponents, while the right hand side has only nonpositive ones. This shows $a_I = 0$ except for the constant term. Therefore $f_0 = f_{|U_0} = \text{const}$. But with the same argument also $f_i = \text{const}$ for all i. As they agree on the overlap, we are done.

4.* Show that the set $\Gamma = \{(z, e^z) : z \in \mathbb{C}\} \subset \mathbb{C}^2$ is not algebraic and determine its closure in the Zariski topology.

Solution

We will show, that the closure $\overline{\Gamma} = V(I(\Gamma))$ of Γ in the Zariski topology is all of $\mathbb{C}^2$. Indeed, assume we have a polynomial

$$P(x,y) = f_n(x)y^n + f_{n-1}(x)y^{n-1} + \ldots + f_1(x)y + f_0(x) \in \mathbb{C}[x,y]$$

on $\mathbb{C}^2$, which vanishes on Γ. Then we have

$$f_n(z)e^{nz} + f_{n-1}(z)e^{(n-1)z} + \ldots + f_0(z) = 0$$

for all $z \in \mathbb{C}$. Dividing by e^{nz} we have

$$f_n(z) + f_{n-1}(z)e^{-z} + \ldots + f_0(z)e^{-nz} = 0.$$

Now for $z \to \infty$ along the positive real axis, the sum $f_{n-1}(z)e^{-z} + \ldots + f_0(z)e^{-nz}$ goes to 0. Hence, also $f_n(z)$ must converge to zero. As $f_n(z)$ is a polynomial however, it must be identically zero for this to be true. Repeating the argument, we show that all $f_{n-1}, \ldots, f_0$ are zero, so $P = 0$. Hence $I(\Gamma) = \{0\}$ and $V(I(\Gamma)) = \mathbb{C}^2$.

Due March 4.