Algebraic Geometry

D-MATH, FS 2016

Prof. Pandharipande

Exercise Sheet 10 - Solutions

1. a) Make the following statement precise and prove it:
"A hypersurface H in $\mathbb{P}^n$ of degree d meets a line l not contained in H in d points, counted with multiplicity."

b) Hypersurfaces of degree d in $\mathbb{P}^n$ are parametrized by $\mathbb{P}(\mathbb{C}[x_0,\dots,x_n]_d)$ (see Exercise 6, Sheet 2) and lines $l \subset \mathbb{P}^n$ are parametrized by $\mathrm{Gr}(2, n+1)$. Show that the set

$$S = \{([f], l) : l \text{ intersects } V(f) \text{ in } d \text{ distinct points}\}$$

contains a nonempty Zariski-open subset of the variety $\mathbb{P}(\mathbb{C}[x_0,\dots,x_n]_d) \times \mathrm{Gr}(2, n+1)$. We say that a line and a degree d hypersurface *in general position* intersect in exactly d points.

Solution

a) A hypersurface H of degree d is given by $H = V(F)$ for $F \in \mathbb{C}[x_0,\dots,x_n]$ homogeneous of degree d. A line l in $\mathbb{P}^n$ is isomorphic to $\mathbb{P}^1$ and we can see it as a morphism

$$\iota : \mathbb{P}^1 \to \mathbb{P}^n, [s,t] \mapsto [a_0 s + b_0 t, \dots, a_n s + b_n t]$$

with image l. Using this form of ι, we see that $\iota([s_0,t_0])$ is contained in H iff

$$G(s,t) = F(a_0 s + b_0 t, \dots, a_n s + b_n t)$$

is zero at $[s,t] = [s_0,t_0]$. But G is a homogeneous polynomial in s,t of degree d (as l is not contained in H, the polynomial F is not identically zero). Hence it has d zeroes on $\mathbb{P}^1$, counted with multiplicities. These zeroes correspond to intersection points in $H \cap l$ and the multiplicities of the intersections are given by the multiplicities of the roots of G.

b) Using the notation from a) we see that the line l corresponds to the point $\mathrm{Span}(\mathbf{a}, \mathbf{b}) \in \mathrm{Gr}(2, n+1)$. Here we can restrict to the chart U_{01} of $\mathrm{Gr}(2, n+1)$ where we set $a_0 = b_1 = 1$ and $a_1 = b_0 = 0$. The remaining variables $a_2,\dots,a_n, b_2,\dots,b_n$ are the coordinates on $U_{01} \cong \mathbb{C}^{2(n-1)}$.

On $\mathbb{P}(\mathbb{C}[x_0,\dots,x_n]_d)$ choose homogeneous coordinates $(c_I)_{I\in\mathbb{N},|I|=d}$ corresponding to the polynomial

$$f = \sum_{I\in\mathbb{N},|I|=d} c_I x^I \in \mathbb{C}[x_0,\dots,x_n]_d.$$

Then writing the polynomial G above as

$$G(s,t) = \lambda_d s^d + \lambda_{d-1}s^{d-1}t + \ldots, +\lambda_0 t^d,$$

the coefficients λ_i depend linearly on the coordinates c_I and algebraically on the coordinates $a_2, \ldots, b_n$. If $\lambda_d \neq 0$, the roots of G on $\mathbb{P}^1$ are exactly the roots of

$$\tilde{G}(s) = \lambda_d s^d + \lambda_{d-1}s^{d-1} + \ldots + \lambda_0$$

for $s \in \mathbb{C}$. The polynomial $\tilde{G}$ has d distinct roots iff the resultant $R = \mathrm{Res}(\tilde{G}, \tilde{G}')$ of $\tilde{G}$ and its derivative $\tilde{G}'$ with respect to s does not vanish. But R is algebraic (and even homogeneous) in the coefficients of $\tilde{G}$ (and thus G).

We conclude that the set S above indeed contains a nonempty Zariski-open subset: Let $U = \{((c_I)_I, (a_2, \ldots, b_n)) : \lambda_d \neq 0\}$ be the complement of the vanishing set of λ_d, which is algebraic in the c_I and the $a_2, \ldots, b_n$. Then the smaller open set $U' = \{((c_I)_I, (a_2, \ldots, b_n)) \in U : R \neq 0\}$ is contained in S. To see that U' is nonempty, construct any point in U', for instance

$$a_2 = \ldots = b_n = 0, f = \prod_{k=1}^{d}(x_1 - kx_0).$$

2. In this exercise, we want to show the following result.

Theorem 1. *Let $C \subset \mathbb{P}^2$ be an irreducible curve of degree d. Then C has at most $\binom{d-1}{2}$ singular points.*

a) Show the Theorem for $d = 1$.

b) For $d = 2$ recall the following result from Sheet 4, Exercise 4:

Lemma 1. *For five points in general linear position in $\mathbb{P}^2$ there exists a rational normal curve $\nu : \mathbb{P}^1 \to \mathbb{P}^2$ passing through them.*

Use this together with the Theorem of Bezout to show that every irreducible conic C in the plane $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$, hence has no singular point as desired. *Note*: This result can also be shown by projection from a point $p \in C$ to a line $l \subset \mathbb{P}^2$ not going through p.

c) Show the Theorem for $d \geq 3$ as follows: assume we have distinct singular points $a_1, \ldots, a_{\binom{d-1}{2}+1}$ of C. Choose additional points $b_i \in C$ and construct a curve C' of degree d' going through all points a_j, b_i. Arrive at a contradiction using Bezout's Theorem. To construct C' use Exercise 6, Sheet 2.

d) Show that a (not necessarily irreducible) curve C in $\mathbb{P}^2$ of degree d has at most $\binom{d}{2}$ singular points. Can you find an example for each d where this number is reached?

Solution

a) For $d = 1$, C is a line and thus isomorphic to $\mathbb{P}^1$, which has no singular points as desired.

b) Given an irreducible conic $C \subset \mathbb{P}^2$, choose p_1, p_2 distinct points on C. Then choose $p_3 \in C$ away from the intersection of C with the line through p_1, p_2. Continue choosing p_4, p_5 such that no three of the points lie on a line. This continues to be possible since C cannot be covered by a finite collection of lines. Indeed, as C is irreducible, this would mean that already a single line l covers it, so $C \subset l$. Then by the Nullstellensatz, the equation of C divides a power of the equation of the line l, so we would have $C = V(h^2)$, where h is homogeneous of degree 1, i.e. C is a "double plane". Thus $C \cong \mathbb{P}^1$ (although we would not want to call such a double plane a conic normally).

Returning to our original construction, we have now five points $p_1, \ldots, p_5 \in C$ in general position. Hence there exists a rational normal curve $\nu : \mathbb{P}^1 \to \mathbb{P}^2$ through them. As we have seen, the image D of such a curve is cut out by a single equation of degree 2 (see Exercise 3, Sheet 4). For the standard rational normal curve $[s,t] \mapsto [s^2, st, t^2]$ this is the quadric $XZ - Y^2$ with coordinates X, Y, Z on $\mathbb{P}^2$.

Now by assumption C, D both have degree 2, but they meet in the five points $p_1, \ldots, p_5$. If they had no irreducible component in common, they would however meet in at most $2 \cdot 2 = 4$ points. Hence they must have a common irreducible component. But we know that C is irreducible and that D is isomorphic to $\mathbb{P}^1$ via ν, thus also irreducible. We conclude that $C = D$ and that $\nu : \mathbb{P}^1 \to C$ is an isomorphism as desired.

c) This proof is done in `http://www.mathematik.uni-kl.de/~gathmann/class/alggeom-2014/chapter-13.pdf`, Proposition 13.5.

d) Let $C = V(F)$ for F a homogeneous polynomial of degree d and write $F = F_1 \cdots F_k$ as a product of irreducible homogeneous polynomials, such that $C = C_1 \cup \ldots \cup C_k$ with $C_i = V(F_i)$ is the decomposition into irreducible components. The singular points of C are given by

$$V(F, \frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_2}).$$

We claim that for each of the components C_i, there is a j such that $\partial F / \partial x_j$ does not vanish on C_i. For convenience let $i = 1$, then

$$\frac{\partial F}{\partial x_j} = \frac{\partial F_1}{\partial x_j} F_2 \cdots F_k + F_1 G,$$

where $G = \partial(F_2 \ldots F_k)/\partial x_j$. We know that F_1 vanishes on C_1, so if $\partial F / \partial x_j$ vanishes on C_1, we have that

$$C_1 = V(F_1) \subset V(\frac{\partial F_1}{\partial x_j} F_2 \cdots F_k).$$

By the Nullstellensatz and as F_1 is irreducible, this means that F_1 divides $\frac{\partial F_1}{\partial x_j} F_2 \cdots F_k$. As by assumption the F_l are all coprime, this means $F_1 | \frac{\partial F_1}{\partial x_j}$, so for degree reasons $\frac{\partial F_1}{\partial x_j} = 0$. But this partial derivative cannot vanish for all j, as otherwise $F_1 = 0$. This proves the statement.

Then a general linear combination

$$G = \lambda_0 \frac{\partial F}{\partial x_0} + \ldots + \lambda_2 \frac{\partial F}{\partial x_2}$$

vanishes on none of the components C_i of C. Hence, by Bezout, C and $V(G)$ intersect in a finite number of points, and this number is bounded from above by $d(d-1)$. Also all of the singular points $q_1, \ldots, q_s$ of C are contained in $C \cap V(G)$ and at the q_j, the intersection multiplicity of F and G is at least 2, as q_j is a multiple point for F. Thus we have $2s \leq d(d-1)$, which proves the desired claim.

This bound is obtained if C is a union of d distinct lines, all meeting pairwise at distinct points. Then the number of intersection points is $\binom{d}{2}$ and these are exactly the singular points of C.

Due May 20.