Algebraic Geometry

D-MATH, FS 2016

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Exercise Sheet 11 - Solutions

1. Define an affine scheme $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$ and show that

a) the stalk at $p \in \mathrm{Spec}(A)$ is A_p, the localization of A along p.

b) for f be an element of A,

$$\mathcal{O}_{\mathrm{Spec}(A)}(D(f)) = A_f.$$

Here $D(f)$ is the open set of primes not containing f and A_f is the localization of A at the element f. In particular $\mathcal{O}_{\mathrm{Spec}(A)}(\mathrm{Spec}(A)) = A$.

Solution Hartshorne, Chp II., Proposition 2.2.

2. Prove that the morphisms from $\mathrm{Spec}(B)$ to $\mathrm{Spec}(A)$ as locally ringed spaces are in bijective correspondence to the ring homomorphisms $A \to B$.

Solution Hartshorne, Chp II., Proposition 2.3.

3. Let $\mathcal{F}$ be a presheaf on a topological space X. For every $x \in X$, let $\mathcal{F}_x$ be the stalk of $\mathcal{F}$ at x. For $x \in U$, U open, let $\rho_{U,x} : \mathcal{F}(U) \to \mathcal{F}_x$ be the induced restriction map. Define a sheaf $\mathcal{F}^{\mathrm{sh}}$, the sheafification of $\mathcal{F}$ or the sheaf *associated to* $\mathcal{F}$, by

$$\begin{aligned}\mathcal{F}^{\mathrm{sh}}(U) = \{(s_x \in \mathcal{F}_x)_{x\in U} \mid \text{ for all } x \in U \text{ there exists an open set } V \text{ with} \\ x \in V \subset U \text{ and } s \in \mathcal{F}(V), \text{ such that } \rho_{V,y}(s) = s_y \text{ for all } y \in V\}\end{aligned}$$

(i) Show that $\mathcal{F}^{\mathrm{sh}}$ is a sheaf.

(ii) Prove that $(\mathcal{F}^{\mathrm{sh}})_x \cong \mathcal{F}_x$ for all $x \in X$.

(iii) Let $f : \mathcal{F} \to \mathcal{F}^{\mathrm{sh}}$ be the natural map given by $s \in \mathcal{F}(U) \mapsto (\rho_{U,x}(s))_{x\in U} \in \mathcal{F}^{\mathrm{sh}}(U)$. Show that it satisfies the following universal property: For any sheaf $\mathcal{G}$ and any map of presheaves $g : \mathcal{F} \to \mathcal{G}$, there exists a unique map $\bar{g} : \mathcal{F}^{\mathrm{sh}} \to \mathcal{G}$ such that $\bar{g} \circ f = g$.

Solution

(i) For $V \subset U$ two open sets, we have a restriction map

$$\rho_{VU} : \mathcal{F}^{\mathrm{sh}}(U) \to \mathcal{F}^{\mathrm{sh}}(V), (s_x \in \mathcal{F}_x)_{x\in U} \mapsto (s_x \in \mathcal{F}_x)_{x\in V}.$$

As the condition on the families (s_x) is local, this map is well-defined. It is clear that ρ_{UU} is the identity and that for $W \subset V \subset U$ we have $\rho_{WV} \circ \rho_{VU} = \rho_{WU}$. This shows that $\mathcal{F}$ is a presheaf.

Let U be an open set in X and assume we have a cover $U = \bigcup_{\alpha \in A} V_\alpha$ of U by open sets V_α. Then giving sections

$$s^\alpha = (s_x^\alpha \in \mathcal{F}_x)_{x \in V_\alpha} \in \mathcal{F}^{\text{sh}}(V_\alpha)$$

such that s^α, s^β restrict to the same section on $V_\alpha \cap V_\beta$ means simply $s_x^\alpha = s_x^\beta$ for all $x \in V_\alpha \cap V_\beta$. But then for every $x \in U$ there is a unique $s_x = s_x^\alpha$ for any $\alpha \in A$ with $x \in V_\alpha$ and we obtain a section $s = (s_x \in \mathcal{F}_x)_{x \in U}$ on all of U. This shows that $\mathcal{F}^{\text{sh}}$ is a sheaf.

(ii) Given $[(U, s)] \in \mathcal{F}_x$, where U is a neighbourhood of x and $s \in \mathcal{F}(U)$, we obtain an element $(\rho_{U,y}(s))_{y \in U}$ of $\mathcal{F}^{\text{sh}}(U)$ and thus $[(U, (\rho_{U,y}(s))_{y \in U})] \in \mathcal{F}_x^{\text{sh}}$. Conversely, let $[(U, (s_y)_{y \in U})] \in \mathcal{F}_x^{\text{sh}}$, then by assumption there exists an open neighbourhood V of x and a section $s \in \mathcal{F}(V)$ with $\rho_{V,y}(s) = s_y$ for $y \in V$. Thus we have an element $[(V, s)] \in \mathcal{F}_x$. These maps are well-defined and give inverse isomorphisms $\mathcal{F}_x \cong \mathcal{F}_x^{\text{sh}}$.

(iii) Assume we are given a sheaf $\mathcal{G}$ and a map $g : \mathcal{F} \to \mathcal{G}$ and we want $\bar{g}$ with $\bar{g} \circ f = g$. Given $x \in X$, this equation implies that we have $\bar{g}_x \circ f_x = g_x$ as the corresponding maps of the stalks $(\mathcal{F}^{\text{sh}})_x \to \mathcal{G}_x$. But from (ii) we know that f_x is an isomorphism, so $\bar{g}_x = g_x \circ (f_x)^{-1}$ is uniquely determined. We define the map $\bar{g}(U) : \mathcal{F}^{\text{sh}}(U) \to \mathcal{G}(U)$ as follows: given $(s_x \in \mathcal{F}_x)_{x \in U} \in \mathcal{F}^{\text{sh}}(U)$ we obtain a family $(t_x = (g_x \circ (f_x)^{-1})(s_x) \in \mathcal{G}_x)_{x \in U}$. We need to show that there is a section $t \in \mathcal{G}(U)$ with stalk t_x at $x \in U$, which will be the image of $(s_x)_{x \in U}$ under $\bar{g}(U)$.

By definition of $\mathcal{F}^{\text{sh}}$, for every $x \in U$ there exists an open neighbourhood V_x of x in U and a section $S_x \in \mathcal{F}(V_x)$ with $s_y = \rho_{V_x,y}(S_x)$. But then the section $T_x = g(V_x)(S_x) \in \mathcal{G}(V_x)$ has stalk t_y at all $y \in V_x$. Thus we have the open cover $U = \bigcup_{x \in U} V_x$ and the elements $T_x \in \mathcal{G}(V_x)$ and they agree on intersections $V_{x'} \cap V_{x''}$ (as their stalks $t_y, y \in V_{x'} \cap V_{x''}$ agree there). As $\mathcal{G}$ is a sheaf, these sections indeed glue to a unique section $T \in \mathcal{G}(U)$.

The maps $\bar{g}(U)$ described above are compatible with restrictions and thus define a sheaf map $\bar{g}$. From the construction, we see that as desired $\bar{g}_x \circ f_x = g_x$. As a map into a sheaf is uniquely determined by its induced map on stalks, we have $\bar{g} \circ f = g$ and we also have that $\bar{g}$ is unique with this property.

Due May 27.