Exercise Sheet 12 - Solutions

1. Given a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of abelian groups on a space X and $U \subset X$ open, let

$$\ker^{\text{pre}}(\varphi)(U) = \ker(\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)),$$
$$\operatorname{coker}^{\text{pre}}(\varphi)(U) = \operatorname{coker}(\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)).$$

a) Describe the natural "restriction maps"

$$\ker^{\text{pre}}(\varphi)(U) \to \ker^{\text{pre}}(\varphi)(V), \operatorname{coker}^{\text{pre}}(\varphi)(U) \to \operatorname{coker}^{\text{pre}}(\varphi)(V)$$

for $V \subset U$ open and show that this data defines presheaves of abelian groups.

b) Prove that for the stalks of the above presheaves we have

$$\ker^{\text{pre}}(\varphi)_p = \ker(\mathcal{F}_p \xrightarrow{\varphi_p} \mathcal{G}_p), \operatorname{coker}^{\text{pre}}(\varphi)_p = \operatorname{coker}(\mathcal{F}_p \xrightarrow{\varphi_p} \mathcal{G}_p).$$

c) Show that $\ker(\varphi) = \ker^{\text{pre}}(\varphi)$ is a sheaf if $\mathcal{F}, \mathcal{G}$ are sheaves.

d) For $X = \mathbb{C}$ let $\mathcal{F} = (\mathcal{O}, +)$ be the sheaf of holomorphic functions (with addition) and $\mathcal{G} = (\mathcal{O}^*, \cdot)$ be the sheaf of nowhere zero holomorphic functions (with multiplication). Then there is a map $\exp : \mathcal{O} \to \mathcal{O}^*$ of sheaves of abelian groups defined by

$$\exp(U) : \mathcal{O}(U) \to \mathcal{O}^*(U), f \mapsto \exp(f).$$

i) Compute $\ker(\exp)$.

ii) Show that $\operatorname{coker}^{\text{pre}}(\exp)$ is not a sheaf.

iii) In general, for a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of sheaves of abelian groups, we define the cokernel of φ as the sheafification

$$\operatorname{coker}(\varphi) = (\operatorname{coker}^{\text{pre}}(\varphi))^{\text{sh}}$$

of $\operatorname{coker}^{\text{pre}}(\varphi)$. Compute $\operatorname{coker}(\exp)$.

Solution

a) By definition, for $V \subset U$ open subsets of X, we have a commutative diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \ker(\varphi(U)) & \longrightarrow & \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) & \longrightarrow & \operatorname{coker}(\varphi(U)) & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \rho_{\mathcal{F}}} & & \downarrow{\scriptstyle \rho_{\mathcal{G}}} & & & & \\ 0 & \longrightarrow & \ker(\varphi(V)) & \longrightarrow & \mathcal{F}(V) & \xrightarrow{\varphi(V)} & \mathcal{G}(V) & \longrightarrow & \operatorname{coker}(\varphi(V)) & \longrightarrow & 0 \end{array}$$

where the rows are exact sequences and the vertical arrows $\rho_{\mathcal{F}}, \rho_{\mathcal{G}}$ are the restriction maps from U to V. There are natural maps $\ker(\varphi(U)) \to \ker(\varphi(V))$ and $\operatorname{coker}(\varphi(U)) \to \operatorname{coker}(\varphi(V))$ to complete the diagram above. Indeed, for $f \in \ker(\varphi(U)) \subset \mathcal{F}(U)$ we have

$$\varphi(V)(\rho_{\mathcal{F}}(f)) = \rho_{\mathcal{G}}(\varphi(U)(f)) = \rho_{\mathcal{G}}(0) = 0,$$

so $\rho_{\mathcal{F}}(f) \in \ker(\varphi(V))$ is the natural restriction of f to V.

On the other hand, for $[g] \in \operatorname{coker}(\varphi(U)) = \mathcal{G}(U)/\varphi(U)(\mathcal{F}(U))$ we want to take $[\rho_{\mathcal{G}}(g)]$ as the restriction to V. To show that this is well-defined, assume we take a different representative $g + \varphi(U)(f)$ of $[g]$. Then

$$\rho_{\mathcal{G}}(g + \varphi(U)(f)) = \rho_{\mathcal{G}}(g) + \varphi(V)(\rho_{\mathcal{F}}(f))$$

is equivalent to $\rho_{\mathcal{G}}(g)$ modulo the image of $\varphi(V)$ as desired.

This finishes the description of the restriction maps. As they were defined using the restriction maps of $\mathcal{F}$ and $\mathcal{G}$ and as these two are presheaves, the restriction maps of $\ker^{\text{pre}}(\varphi)$, $\operatorname{coker}^{\text{pre}}(\varphi)$ satisfy the natural compatibility conditions.

b) As $\ker^{\text{pre}}(\varphi)(U) \subset \mathcal{F}(U)$ for all U, we naturally have $\ker^{\text{pre}}(\varphi)_p \subset \mathcal{F}_p$. But an element $[(U, f)] \in \mathcal{F}_p$ maps to $[(U, \varphi(U)(f))] \in \mathcal{G}_p$. This is zero iff there exists an open neighbourhood V of p in U with $\varphi(U)(f)|_V = 0$. But then $f|_V \in \ker(\varphi(V))$, so $[(U, f)] = [(V, f|_V)] \in \ker^{\text{pre}}(\varphi)_p$.

For the cokernel, we have that elements of $\operatorname{coker}^{\text{pre}}(\varphi)_p$ are $[(U, [g])]$, where U is an open neighbourhood of p and $[g] \in \mathcal{G}(U)/\varphi(U)(\mathcal{F}(U))$. On the other hand, elements of $\operatorname{coker}(\varphi_p)$ are $[[(V, h)]] \in \mathcal{G}_p/\varphi_p(\mathcal{F}_p)$. The natural map $[(U, [g])] \mapsto [[(U, g)]]$ is well-defined and surjective. It is also injective, as $[[(U, g)]] = 0$ iff there exists $p \in V \subset U, f \in \mathcal{F}(V)$ with $g|_V = \varphi(V)(f)$ and then $[(U, [g])] = [(V, [g|_V])] = 0$.

c) For $f \in \ker(\varphi)(U)$ and an open cover $U = \bigcup_i U_i$ with $f|_{U_i} = 0$ for all i, we have $f = 0$ as $\mathcal{F}$ is a sheaf and as the restriction of f is defined by the restriction in $\mathcal{F}$.

For an open cover $U = \bigcup_i U_i$ and elements $f_i \in \ker(\varphi)(U_i)$ with $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all i, j, we know that we can glue the f_i to some element $f \in \mathcal{F}(U)$, again because $\mathcal{F}$ is a sheaf. To show that $f \in \ker(\varphi)(U)$ let $g = \varphi(U)(f)$. Then $g|_{U_i} = \varphi(U_i)(f|_{U_i}) = 0$ for all i. So as $\mathcal{G}$ is a sheaf, we have $g = 0$ as desired.

d) i) Let $U \subset X$ be open and $f \in \mathcal{O}(U)$. Then $\exp(f) = 1$ iff f has image in $2\pi i\mathbb{Z}$ (in particular, f is locally constant). Conversely any continuous function with values in $2\pi i\mathbb{Z}$ is automatically holomorphic. Therefore $\ker(\exp)$ is the sheaf $2\pi i\underline{\mathbb{Z}}$ of locally constant functions with values in $2\pi i\mathbb{Z}$.

ii) Let $U = \mathbb{C} \setminus \{0\}$, which is covered by the open sets $U_1 = \mathbb{C} \setminus [0, \infty)$ and $U_2 = \mathbb{C} \setminus (-\infty, 0]$. By complex analysis, we know that the function $g = z \in \mathcal{O}^*(U)$ cannot be written as the exponential of some other holomorphic function $f \in \mathcal{O}(U)$. Thus $[g] \neq 0 \in \operatorname{coker}(\exp(U))$. On the other hand, the open sets U_1, U_2 are simply connected, so every nowhere zero function $\tilde{g}$ on them can be written as $\exp(\tilde{f})$. Thus $\mathrm{coker}(\exp(U_i)) = 0$ for $i = 1, 2$. But thus $\mathrm{coker}^{\mathrm{pre}}(\exp)$ cannot be a sheaf, because the restriction of g to the open cover U_1, U_2 of U is zero on both sets, but globally nonzero.

iii) We will show that the stalk of $\mathrm{coker}^{\mathrm{pre}}(\exp)$ at all points $p \in X$ is zero. Then by the construction of the sheafification, $\mathrm{coker}(\exp) = 0$.
Given $p \in X$ and U an open neighbourhood with a nowhere zero holomorphic function g on U, we want to show $[(U, g)] = 0 \in \mathrm{coker}^{\mathrm{pre}}(\exp)_p$. As U is open, there exists $r > 0$ such that the open ball $B_r(p)$ is contained in U. Then $[(U, g)] = [(B_r(p), g|_{B_r(p)})]$, but as above, the restriction of g now has a logarithm on the simply connected domain $B_r(p)$. Thus g is contained in the image of $\exp(B_r(p))$ and thus $[(U, g)] = 0$ as desired.

From this, we obtain the so-called *exponential exact sequence*

$$0 \to 2\pi i \underline{\mathbb{Z}} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0$$

for $X = \mathbb{C}$. This is an exact sequence in more general cirumstances, for instance on complex manifolds X.

2. a) Let $\mathcal{F}, \mathcal{G}$ be sheaves on a topological space X. Show that a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ is an isomorphism if and only if the induced maps $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ on the stalks at all points $p \in X$ are isomorphisms.

b) Let $\mathcal{F}, \mathcal{F}', \mathcal{F}''$ be sheaves of abelian groups on X and assume we have a sequence

$$\mathcal{F}' \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}''$$

such that $\psi \circ \varphi = 0$. The sequence is called *exact at* $\mathcal{F}$ if the natural map

$$\mathrm{im}(\varphi) = \ker(\mathcal{F} \to \mathrm{coker}(\varphi)) \to \ker(\psi)$$

is an isomorphism. Show that this is equivalent to the condition that the sequence

$$\mathcal{F}'_p \xrightarrow{\varphi_p} \mathcal{F}_p \xrightarrow{\psi_p} \mathcal{F}''_p$$

of maps induced on the stalks is exact for all $p \in X$.

Solution

a) It is clear that if φ is an isomorphism, all maps φ_p are isomorphisms. Now assume the φ_p are isomorphisms and let $U \subset X$ be open. Then we need to show that $\mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism.
For injectivity, assume we have $f, f' \in \mathcal{F}(U)$ with $g = \varphi(U)(f) = \varphi(U)(f')$. Then for all $p \in X$ we have $\varphi_p([(U, f)]) = [(U, g)] = \varphi_p([(U, f')]) \in \mathcal{G}_p$, so $[(U, f)] = [(U, f')] \in \mathcal{F}_p$. By definition this means every point p has a neighbourhood U_p in U such that $f|_{U_p} = f'|_{U_p}$. By the uniqueness part of the sheaf axioms of $\mathcal{F}$ this implies $f = f'$.
For surjectivity, let $g \in \mathcal{G}(U)$ then for all p there exists a neighbourhood U_p and $f_p \in \mathcal{F}(U_p)$ with $\varphi_p([(U_p, f_p)]) = [(U_p, \varphi(U_p)(f_p))] = [(U, g)] \in \mathcal{G}_p$.

By shrinking U_p if necessary, we may thus assume that $\varphi(U_p)(f_p) = g|_{U_p}$. Then the sections $f_p, f_{p'}$ agree on $U_p \cap U_{p'}$, because their stalks at all points $q \in U_p \cap U_{p'}$ agree (they are the unique preimage of g_q under φ_p). By the sheaf axioms of $\mathcal{F}$ these sections glue to a section $f \in \mathcal{F}(U)$. Now f maps to some $g' \in \mathcal{G}(U)$ by $\varphi(U)$. But by construction, g and g' agree on the open cover U_p of X, so $g = g'$ as desired.

b) Note first that as $\psi \circ \varphi = 0$, the map $\psi : \mathcal{F} \to \mathcal{F}''$ factors through $\operatorname{coker}^{\mathrm{pre}}(\varphi)$ in a natural way. But by the universal property of sheafification, it must then also factor through $\operatorname{coker}(\varphi)$, so we have a sequence

$$\mathcal{F}' \xrightarrow{\varphi} \mathcal{F} \to \operatorname{coker}(\varphi) \xrightarrow{\bar{\psi}} \mathcal{F}''.$$

But then $\operatorname{im}(\varphi) = \ker(\mathcal{F} \to \operatorname{coker}(\varphi))$ naturally sits inside $\ker(\psi)$. This gives the natural map above.

By the first exercise part it is an isomorphism iff the corresponding map of stalks is an isomorphism for all points p of X. But by Exercise 1 b) this is exactly the map

$$\ker(\mathcal{F}_p \to \operatorname{coker}(\varphi_p)) \to \ker(\psi_p),$$

which by basic algebra is an isomorphism iff $\mathcal{F}'_p \to \mathcal{F}_p \to \mathcal{F}''_p$ is exact at $\mathcal{F}_p$.

3. Let X be a topological space and let $\mathcal{U} = (U_\alpha)_{\alpha \in A}$ be a base of the topology of X. A *sheaf F on the base $\mathcal{U}$* is a collection $(F(U_\alpha))_{\alpha \in A}$ of sets together with morphisms

$$\rho_{\beta\alpha} : F(U_\alpha) \to F(U_\beta)$$

for $U_\beta \subset U_\alpha$, such that $\rho_{\alpha\alpha} = \mathrm{id}$ and

$$\rho_{\gamma\beta} \circ \rho_{\beta\alpha} = \rho_{\gamma\alpha}$$

for $U_\gamma \subset U_\beta \subset U_\alpha$. Moreover, for $U_\alpha = \bigcup_{\beta \in B} U_\beta$ and elements $f_\beta \in F(U_\beta)$ such that $\rho_{\gamma\beta}(f_\beta) = \rho_{\gamma\beta'}(f_{\beta'})$ for all β, β', γ with $U_\gamma \subset U_\beta \cap U_{\beta'}$ there exists a unique $f_\alpha \in F(U_\alpha)$ such that $\rho_{\beta\alpha}(f_\alpha) = f_\beta$ for $\beta \in B$.

For a sheaf F on the base $\mathcal{U}$ and $p \in X$ define

$$F_p = \varinjlim_{U_\alpha \ni p} F(U_\alpha).$$

a) Show that the data

$$\mathcal{F}(U) = \left\{ (f_p \in F_p)_{p \in U} : \begin{array}{l} \text{for all } p \in U, \text{ there exists } U_\alpha \ni p,\ s \in F(U_\alpha) \\ \text{with } s_q = f_q \text{ for all } q \in U_\alpha \end{array} \right\}$$

defines a sheaf on X.

b) Show that the natural map $F(U_\alpha) \to \mathcal{F}(U_\alpha), f \mapsto (f_p)_{p \in U_\alpha}$ is an isomorphism for all $\alpha \in A$.

c) Prove that for any other sheaf $\mathcal{G}$ on X with isomorphisms $\mathcal{G}(U_\alpha) \cong F(U_\alpha)$ compatible with the restriction maps on both sides, we have $\mathcal{F} \cong \mathcal{G}$ (so $\mathcal{F}$ is the unique sheaf with this property, up to isomorphism). Conclude that for a ring R, we have $\tilde{R} = \mathcal{O}_{\mathrm{Spec}(R)}$.

Solution

a) This can be shown similar as Exercise 3 (i) on Sheet 11.

b) To show that $F(U_\alpha) \to \mathcal{F}(U_\alpha)$ is injective, assume we have $s, s' \in F(U_\alpha)$ with $s_p = s'_p$ for all $p \in U_\alpha$. Then as the U_α form a base of the topology, for every $p \in U_\alpha$ there exists $\alpha_p \in A$ with $p \in U_{\alpha_p} \subset U_\alpha$ and $s|_{U_{\alpha_p}} = s'|_{U_{\alpha_p}}$. But the U_{α_p} cover U_α, so we must have $s = s'$ (here we use the uniqueness part of the definition of a sheaf on a base).

To show surjectivity of $F(U_\alpha) \to \mathcal{F}(U_\alpha)$, assume we are given an element $(f_p \in F_p)_{p \in U_\alpha}$ of $\mathcal{F}(U_\alpha)$. Then for every $p \in U_\alpha$ there exists an open neighbourhood U_{α_p} in U and a section $s_p \in F(U_{\alpha_p})$ with $f_q = (s_p)_q$ for all $q \in U_{\alpha_p}$. Again the U_{α_p} form a cover of U_α and the sections s_p agree on overlaps. Indeed, for $p, p' \in U_\alpha$ and $\beta \in A$ with $U_\beta \subset U_{\alpha_p} \cap U_{\alpha_{p'}}$ we have that $s_p|_{U_\beta} = s'_p|_{U_\beta}$, because both sections have the same stalks at the points of U_β (together with the uniqueness part we already showed). Then by definition, we have a section $s \in F(U_\alpha)$ with stalk $s_p = f_p$ as desired.

c) The isomorphisms $\mathcal{G}(U_i) \cong F(U_i)$ induce isomorphisms $\mathcal{G}_p \cong F_p$ as they are compatible with the restriction maps and as the stalk at p can be computed on a basis of open neighbourhoods of p. By the universal property of sheafification, $\mathcal{G}^{\mathrm{sh}} \cong \mathcal{G}$ (as $\mathcal{G}$ is already a sheaf). But by definition, for $U \subset X$ open, we have

$$\mathcal{G}^{\mathrm{sh}}(U) = \left\{ (g_p \in \mathcal{G}_p)_{p \in U} : \begin{array}{l} \text{for all } p \in U, \text{ there exists } V \ni p,\ s \in \mathcal{G}(V) \\ \text{with } s_q = g_q \text{ for all } q \in V \end{array} \right\}.$$

But $\mathcal{G}_p \cong F_p$ and the open sets V can be chosen to be of the form $V = U_\alpha$ such that $s \in \mathcal{G}(U_\alpha) \cong F(U_\alpha)$. But then we have exactly recovered the definition of $\mathcal{F}$ above, so indeed $\mathcal{G} = \mathcal{F}$.

Given a ring R, we have for $f \in R$ that $\tilde{R}(D(f)) = R_f$. But as we have seen before, we also have $\mathcal{O}_{\mathrm{Spec}(R)}(D(f)) = R_f$. Moreover, in both cases the restriction maps from $D(g)$ to $D(f)$ are the maps $R_g \to R_f$ induced by the identity on R. Thus as $\mathcal{O}_{\mathrm{Spec}(R)}$ is a sheaf, it must be $\tilde{R}$ by what we have just proved.

Due June 03.