Algebraic Geometry

D-MATH, FS 2016

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Exercise Sheet 2 - Solutions

1. Prove that the radical of a homogeneous ideal in $\mathbb{C}[x_0, ..., x_n]$ is homogeneous.

Solution

Let $S = \mathbb{C}[x_0, \ldots, x_n]$ and consider a homogeneous ideal $I = \oplus_{d \geq 0} I_d$, with I_d being its homogeneous parts. Let $f = \sum_{d \geq 0} f_d \in \sqrt{I}$ an element of the radical of I with only finitely many f_d non-zero. We need to show that $f_d \in \sqrt{I}$ for all d. We do this by induction on $n = \max\{d | f_d \neq 0\}$. The case $n = 0$ is clear, because then $f = f_0 \in \sqrt{I}$ as desired.

Let $k > 0$ such that $f^k \in I$. Then $f^k = g + f_n^k$ with g of degree strictly smaller than $\deg(f_n^k) = kn$. Hence $f_n^k \in I$ by homogeneity and therefore $f_n \in \sqrt{I}$. But now $f - f_n \in \sqrt{I}$ and therefore by induction $f_d \in \sqrt{I}$ for all d and we are done.

2. Let $V \subset \mathbb{C}^n$ be an affine algebraic variety and let $A = \mathbb{C}[x_1, \ldots, x_n]/I(V)$ be the ring of regular functions on V. Let $g \in A$ and let $V_g \subset V$ be the open set defined by

$$V_g = \{p \in V \mid g(p) \neq 0\}$$

a) Show that, V_g is isomorphic to an affine algebraic variety.

b) What is the ring of regular functions on V_g? (Hint: It is the localization of A at g. Prove this)

Solution

Consider $\mathbb{C}^{n+1}$ with the coordinates given by $x_1, \ldots, x_n, y$ and define $W \subset \mathbb{C}^{n+1}$ by the set of equations $I(V)$ and $g(x_1, \ldots, x_n)y - 1 = 0$. Here we consider $I \subset \mathbb{C}[x_1, \ldots, x_n] \subset \mathbb{C}[x_1, \ldots, x_n, y]$. This defines W as an affine algebraic variety with coordinate ring

$$\mathbb{C}[x_1, \ldots, x_n, y]/(I, yg - 1) \cong \mathbb{C}[x_1, \ldots, x_n, 1/g]/I = A_g.$$

Let $\pi : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^n$ be the projection that forgets the y-coordinate and define $\varphi : V_g \longrightarrow \mathbb{C}^{n+1}$ by $\varphi(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 1/g(x_1, \ldots, x_n))$. Then it is easy to see that the image of φ is W and that φ and π are inverse to each other and thus induce an isomorphism $V_g \cong W$. In particular, their coordinate rings coincide. Thus, the ring of regular functions on V_g is A_g.

3. Determine the ring of algebraic functions on $\mathbb{C}^2 \setminus \{0\}$. Show that it is a quasi-projective variety which is not isomorphic to an affine algebraic variety.

Solution

First notice that $X = \mathbb{C}^2 \setminus \{(0,0)\}$ is covered by the sets $\mathbb{C}^2_x = \{(x,y) \in \mathbb{C}^2 \mid x \neq 0\}$ and $\mathbb{C}^2_y = \{(x,y) \in \mathbb{C}^2 \mid y \neq 0\}$. We can identify regular functions on X with regular functions on $\mathbb{C}^2_x$ and $\mathbb{C}^2_y$ that coincide on their common domain of definition, namely $\mathbb{C}^2_{xy} = \mathbb{C}^2_x \cap \mathbb{C}^2_y$. By the previous exercise, we have

$$\Gamma(\mathbb{C}^2_x) = \mathbb{C}[x, y, 1/x],$$
$$\Gamma(\mathbb{C}^2_y) = \mathbb{C}[x, y, 1/y],$$
$$\Gamma(\mathbb{C}^2_{xy}) = \mathbb{C}[x, y, 1/xy].$$

Let $f/x^n \in \mathbb{C}[x, y, 1/x], g/y^m \in \mathbb{C}[x, y, 1/y]$ with $f, y \in \mathbb{C}[x, y]$ such that x does not divide f and y does not divide g. Assume that $f/x^n = g/y^m \in \mathbb{C}[x, y, 1/xy]$, then we must have $fy^m = gx^n$. But as x does not divide f nor y, we have $n = 0$ and similarly $m = 0$. Hence the regulars functions on X are given by $f = g \in \mathbb{C}[x, y]$.

Now for an affine algebraic variety X over an algebraically closed field, there is a one-to-one correspondence between maximal ideals in the coordinate ring (= the ring of regular functions on X) and the points of X. For $X = \mathbb{C}^2 \setminus \{(0,0)\}$, the coordinate ring is given by $\mathbb{C}[x, y]$. Furthermore the points $P = (a_1, a_2)$ correspond to the maximal ideals $(x - a_1, y - a_2)$ for $(a_1, a_2) \neq (0,0)$. But there is *no* point in X that corresponds to the maximal ideal (x, y), hence X is not affine.

4. For $n, m \geq 1$, let $\mathbb{P}^{nm-1}$ be viewed as the projective space of $n \times m$-matrices. Prove that the locus of matrices of rank exactly k is a quasi-projective variety, denoted $R_k \subset \mathbb{P}^{nm-1}$.

Solution

This problem follows basically from the following two facts:

1. A $m \times n$ matrix M has rank $\leq k$ if and only if all $(k+1) \times (k+1)$ minors of M vanish. Hence the locus of matrices of rank at most k is a projective variety X.
2. A $m \times n$ matrix M has rank $\geq k$ if and only if there exist a $k \times k$ minor of M that does not vanish. This shows that the locus of matrices of rank exactly k is a union of open subsets of X.

5. Recall that $\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\}) / \sim$ with $v \sim \lambda v$ for $\lambda \in \mathbb{C}^*$. Thus every element of $\mathbb{P}^n$ can be represented as $[v]$ for $v \in \mathbb{C}^{n+1} \setminus \{0\}$.

a) Show that the group GL_{n+1} of invertible $(n+1) \times (n+1)$-matrices acts on $\mathbb{P}^n$ by $A[v] = [Av]$ for $A \in \mathrm{GL}_n$, $v \in \mathbb{C}^{n+1} \setminus \{0\}$.

b) A finite set $S \subset \mathbb{P}^n$ is called *in general linear position* if for all distinct $[v_1], \ldots, [v_k] \in S$ with $k \leq n+1$ we have that $v_1, \ldots, v_k$ are linearly independent. Show that for $p_1, \ldots, p_{n+2} \in \mathbb{P}^n$ in general linear position, there is $A \in \mathrm{GL}_{n+1}$ with

$$Ap_1 = [e_1], Ap_2 = [e_2], \ldots, Ap_{n+1} = [e_{n+1}], Ap_{n+2} = [e_1 + \ldots + e_{n+1}],$$

where $e_1, \dots, e_{n+1}$ are the basis vectors of $\mathbb{C}^{n+1}$. Show that A is unique up to scaling by elements in $\mathbb{C}^*$.

Solution

a) First we see that $A[v]$ is well-defined. Indeed, for $v \neq 0$ we have $Av \neq 0$ as A is invertible and furthermore $A[\lambda v] = [\lambda Av] = [Av]$. To show that we have a group action, we remark that $I_{n+1}[v] = [I_{n+1}v] = [v]$ and for $A, B \in \mathrm{GL}_n$ we have $A(B[v]) = A[Bv] = [ABv] = (AB)[v]$.

b) Let $p_i = [v_i]$ for $v_1, \dots, v_{n+2} \in \mathbb{C}^{n+1}$ such that any $n+1$ of the v_i are linearly independent. Using that the first $n+1$ form a basis, we find a unique $\tilde{A} \in \mathrm{GL}_{n+1}$ sending v_i to e_i for $i = 1, \dots, n$. The remaining element v_{n+2} is sent to some vector

$$\tilde{A}v_{n+2} = (b_1, \dots, b_{n+1}).$$

Now we claim that all b_i are nonzero. Indeed, if for instance $b_1 = 0$ then $e_2 = \tilde{A}v_2, \dots, e_{n+1} = \tilde{A}v_{n+1}, \tilde{A}v_{n+2}$ would be linearly dependent (as they all have vanishing first coordinate). But then $v_2, \dots, v_{n+2}$ were already linearly dependent, a contradiction to our assumption. Now modify the matrix $\tilde{A}$ by multiplying the ith row with $1/b_i$ to obtain a new matrix A. Then $Av_i = e_i/b_i$ for $i = 1, \dots, n+1$ and $Av_{n+2} = e_1 + \ldots + e_{n+1}$. This immediately implies that A has the desired property.
Now assume A' is another matrix as in the exercise, then $A'A^{-1}$ fixes the points $[e_1], \dots, [e_{n+1}], [e_1 + \ldots + e_{n+1}]$. Fixing the first $n+1$ points means that $A'A^{-1}$ must be a diagonal matrix and fixing the last point means that all entries of the diagonal must be equal to some $\lambda \in \mathbb{C}^*$. Hence we have $A'A^{-1} = \lambda I_{n+1}$ or $A' = \lambda A$.

6. Let $f \in \mathbb{C}[x_0, \dots, x_n]$ be a homogeneous polynomial of degree $m > 0$. Then its vanishing set $V(f) \subset \mathbb{P}^n$ is called a *hypersurface of degree* m in projective space.

a) Show that the homogeneous polynomials $\mathbb{C}[x_0, \dots, x_n]_m$ of degree m form a vector subspace of dimension $\binom{m+n}{n}$.

b) For any subset $S \subset \mathbb{P}^n$ with at most $\binom{m+n}{n} - 1$ elements, show that there is a hypersurface of degree m containing them.

c) In the case of conics ($m = 2$) in the projective plane ($n = 2$), show that through any 5 points in general linear position there is a unique hypersurface of degree 2.

Solution

a) As the sum and scalar multiples of homogeneous polynomials of degree m are still homogeneous of degree m, the space $\mathbb{C}[x_0, \dots, x_n]_m$ clearly forms a vector subspace of $\mathbb{C}[x_0, \dots, x_n]$. Moreover, every element of this space can uniquely be written as a $\mathbb{C}$-linear combination of monomials $x_0^{\alpha_0} x_1^{\alpha_1} \dots x_n^{\alpha_n}$ with $\alpha_0, \dots, \alpha_n \in \mathbb{Z}_{\geq 0}$ and

$$\alpha_0 + \ldots + \alpha_n = m. \tag{1}$$

Thus these monomials form a basis. To calculate their number, we have to compute the number of partitions as in (1) and we claim that it is $\binom{m+n}{n}$. This can be done by induction on $l = n + m$. The case $l = 0$ is trivial, as then $\alpha_0 = 0$ is the only possibility. Assume we have shown the claim up to $l - 1$ and assume we have a partition $\alpha_0 + \ldots + \alpha_n = m$ with $n + m = l$. Then there are two cases:

- If $\alpha_n = 0$, then $\alpha_0 + \ldots + \alpha_{n-1} = m$ is a partition with $n - 1 + m \leq l$, so we have $\binom{m+n-1}{n-1}$ choices for $\alpha_0, \ldots, \alpha_{n-1}$.
- If $\alpha_n > 0$, then $(\alpha_0 - 1) + \ldots + \alpha_{n-1} = m - 1$ is a partition with $n + m - 1 \leq l$, so we have $\binom{m+n-1}{n}$ choices for $\alpha_0 - 1, \ldots, \alpha_n$.

Together, we have $\binom{m+n-1}{n-1} + \binom{m+n-1}{n} = \binom{m+n}{n}$ choices, as claimed.

b) For each point $p = [v] \in \mathbb{P}^n$, the set $J(p) \subset \mathbb{C}[x_0, \ldots, x_n]_m$ of polynomials f with $f(p) = 0$ is a linear subspace of codimension 1. Indeed, the function

$$\mathbb{C}[x_0, \ldots, x_n]_m \to \mathbb{C}, f \mapsto f(v)$$

is linear, surjective (if $v_i \neq 0$, then $x_i^m(v_i) = v_i^m \neq 0$) and $J(p)$ is its kernel. Thus, for $S \subset \mathbb{P}^n$ with at most $\binom{m+n}{n} - 1$ elements, we have $J(S) = \bigcap_{p \in S} J(p)$ is a linear space of codimension at most $\binom{m+n}{n} - 1$, thus of dimension at least 1. Hence, there is a nonzero $f \in J(S)$ and by definition, it satisfies $f(p) = 0$ for all $p \in S$. This means exactly $S \subset V(f)$.

c) As $\binom{2+2}{2} = 6$, we know that there is at least one f, homogeneous of degree 2, vanishing on the points in S. Now we show that this f is unique up to scaling, meaning that $V(f)$ is the only hypersurface through the five points $p_0, \ldots, p_4$.
An important simplification is that we can assume $p_0 = [1, 0, 0], p_1 = [0, 1, 0], p_2 = [0, 0, 1], p_3 = [1, 1, 1]$ using the previous exercise. Indeed we know that there is an invertible 3×3-matrix A such that Ap_i are the points above for $i = 0, \ldots, 3$. But if $f(x_0, x_1, x_2)$ is a homogeneous polynomial of degree 2 vanishing on the p_i, then $f(A^{-1}(x_0, x_1, x_2)^T)$ is a homogeneous polynomial vanishing on the points $[1, 0, 0], \ldots, [1, 1, 1], Ap_4$.
Now in the basis mentioned above, the polynomial f has a representation

$$f(x_0, x_1, x_2) = ax_0^2 + bx_0x_1 + cx_0x_2 + dx_1^2 + ex_1x_2 + fx_2^2.$$

The fact that f vanishes on $[1, 0, 0], [0, 1, 0], [0, 0, 1]$ is equivalent to $a = d = f = 0$. Thus f has now the form

$$f(x_0, x_1, x_2) = bx_0x_1 + cx_0x_2 + ex_1x_2 = (x_0x_1, x_0x_2, x_1x_2) \cdot \begin{pmatrix} b \\ c \\ e \end{pmatrix}.$$

To show that the two remaining conditions, coming from $[1, 1, 1]$ and $p_4 = [z_0, z_1, z_2]$ are independent, we need to show that the vectors $(1, 1, 1)$ and (z_0z_1, z_0z_2, z_1z_2) are linearly independent.
If $(z_0z_1, z_0z_2, z_1z_2) = (0, 0, 0)$, then two of z_0, z_1, z_2 must vanish. This gives a contradiction, as then for instance $p_4 = [z_0, 0, 0] = [1, 0, 0] = p_0$.

If $(z_0 z_1, z_0 z_2, z_1 z_2) \neq (0,0,0)$, we may rescale (z_0, z_1, z_2) such that $(z_0 z_1, z_0 z_2, z_1 z_2) = (1,1,1)$. Then we see immediately $z_1 = 1/z_0 = z_2 = z_0$, hence $[z_0, z_1, z_2] = [1,1,1]$, a contradiction. This finishes the proof that the conditions coming from $p_0, \ldots, p_4$ are independent and that thus f is unique up to scaling.

Due March 11.