Algebraic Geometry

D-MATH, FS 2016

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Exercise Sheet 4 - Solutions

1. a) Develop a theory of algebraic subvarieties of $\mathbb{P}^n \times \mathbb{P}^m$ using bihomogeneous polynomials. For this you must define the Zariski topology and regular functions.

b) Prove that the projections $\mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^n$ and $\mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^m$ are algebraic maps.

c) The set $\mathbb{P}^n \times \mathbb{P}^m$ is covered by the sets $U_i \times U_j \cong \mathbb{C}^n \times \mathbb{C}^m = \mathbb{C}^{n+m}$ for $i = 0, \ldots, n$, $j = 0, \ldots, m$. Verify that a set $X \subset \mathbb{P}^n \times \mathbb{P}^m$ is Zariski-closed if and only if $X \cap U_i \times U_j \subset \mathbb{C}^{n+m}$ is Zariski-closed for all i, j. For such X, verify that a function $f : V \to \mathbb{C}$ from an open subset $V \subset X$ is algebraic if and only if its restriction to $V \cap U_i \times U_j$ is algebraic for all i, j.

Solution

a) Let $x_0, \ldots, x_n, y_0, \ldots, y_m$ be variables. For $\alpha \in (\mathbb{Z}_{\geq 0})^{n+1}, \beta \in (\mathbb{Z}_{\geq 0})^{m+1}$ we write

$$\mathbf{x}^\alpha = x_0^{\alpha_0} x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \mathbf{y}^\beta = y_0^{\beta_0} y_1^{\beta_1} \cdots y_m^{\beta_m}.$$

A polynomial $F \in \mathbb{C}[x_0, \ldots, x_n, y_0, \ldots, y_m] = \mathbb{C}[\mathbf{x}, \mathbf{y}]$ is called *bihomogeneous of bidegree* (d, e) if it is a $\mathbb{C}$-linear combination of terms $\mathbf{x}^\alpha \mathbf{y}^\beta$ with $|\alpha| = \alpha_0 + \ldots + \alpha_n = d, |\beta| = e$. Such a polynomial has a well-defined zero-locus $V(F) \subset \mathbb{P}^n \times \mathbb{P}^m$.

An algebraic variety $X \subset \mathbb{P}^n \times \mathbb{P}^m$ is the common zero locus of finitely many bihomogeneous polynomials $F_1, \ldots, F_k \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$. The open sets of the Zariski topology on X are given by sets of the form

$$X \setminus (V(G_1) \cup V(G_2) \cup \ldots \cup V(G_r)),$$

for $G_1, \ldots, G_r \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$ bihomogeneous. A function $f : U \to \mathbb{C}$ on an open subset $U \subset X$ is algebraic if for every point $p \in U$ there is a neighborhood W of p in U and bihomogeneous polynomials h, g of the same bidegree with $g \neq 0$ on W such that $f|_W = h/g$.

As before, a map $f : U \to W$ (where U, W are either quasi-projective varieties or open subsets of a variety in $\mathbb{P}^n \times \mathbb{P}^m$) is algebraic if it is continuous with respect to the Zariski topology and respects composition with algebraic functions.

b) The projection $\pi_1 : \mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^n$ is continuous as the preimage of $V(F)$ for $F \in \mathbb{C}[\mathbf{x}]$ homogeneous of degree d is exactly $V(F)$, where now $F \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$ is thought of as a bihomogeneous polynomial of bidegree $(d, 0)$.

We have seen before that we can check that a continuous function between quasi-projective varieties is algebraic by restricting to an open cover of the target space. By the same arguments as before, we can also apply this in our situation (where the domain is now $\mathbb{P}^n \times \mathbb{P}^m$). The target $\mathbb{P}^n$ is covered by the sets U_i, $i = 0, \dots, n$ as usual. Here, our morphism is given by

$$\pi_1 : U_i \times \mathbb{P}^m \to U_i \cong \mathbb{C}^n$$
$$([x_0, \dots, x_n], [y_0, \dots, y_m]) \mapsto \left(\frac{x_0}{x_i}, \dots, \widehat{\frac{x_i}{x_i}}, \dots, \frac{x_n}{x_i} \right).$$

We see that all coordinate functions are the quotients of bihomogeneous polynomials x_j, x_i of bidegree $(1, 0)$ and thus algebraic functions. Thus π_1 is algebraic.

c) The important point for us is to relate bihomogeneous polynomials in $x_0, \dots, x_n, y_0, \dots, y_m$ with arbitrary polynomial in $x_0, \dots, \widehat{x_i}, \dots, \widehat{y_j}, \dots, y_m$. We do this for $i = j = 0$. For any bihomogeneous F, we have that $f = F^{uh} = F(1, x_1, \dots, x_n, 1, y_1, \dots, y_m)$ is a polynomial. Conversely, for some polynomial $f \in \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_m]$ of maximal total degree d in the x_i and e in the y_j, we can homogenize it with respect to x_0, y_0 by multiplying the monomial $x_1^{\alpha_1} \dots x_n^{\alpha_n} y_1^{\beta_1} \dots y_m^{\beta_m}$ with $x_0^{d - \sum_{i=1}^n \alpha_i} y_0^{e - \sum_{j=1}^m \beta_j}$. Thus we obtain a polynomial $F = f^h$ which is homogeneous of bidegree (d, e).
For F a bihomogeneous polynomial, we have $V(F) \cap U_{00}$ is cut out by the polynomial F^{uh} in U_{00}. Conversely, for $V(f) \subset U_{00}$ closed we have $V(f) = V(f^h) \cap U_{00}$, so $V(f)$ is closed in the subset topology of $U_{00} \subset \mathbb{P}^n \times \mathbb{P}^m$. This shows that the usual topology on $U_{ij} \cong \mathbb{C}^{nm}$ is the subset topology for $U_{ij} \subset \mathbb{P}^n \times \mathbb{P}^m$. The claimed result follows as a subset $C \subset X$ of a topological space X with an open cover $X = \bigcup_\alpha U_\alpha$ is closed if and only if $C \cap U_\alpha \subset U_\alpha$ is closed for all α.
Similarly, if a regular function is locally given by F/G for F, G bihomogeneous of the same bidegree, then on U_{00} it is given by F^{uh}/G^{uh}. Conversely, given f, g polynomials in $x_1, \dots, x_n, y_1, \dots, y_n$, we can homogenize them with x_0, y_0 to have the same bidegrees and then

$$\frac{f}{g} = \frac{f^h}{g^h}\Big|_{U_{00}}.$$

2. For $n, m \geq 1$ integers, the *Segre-embedding* of $\mathbb{P}^n \times \mathbb{P}^m$ in $\mathbb{P}(\mathbb{C}^{n+1} \otimes \mathbb{C}^{m+1}) = \mathbb{P}^{(n+1)(m+1)-1}$ is the map

$$\sigma : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1}$$
$$([x_0, \dots, x_n], [y_0, \dots, y_m]) \mapsto \left[(x_i y_j)_{\substack{i=0,\dots,n \\ j=0,\dots,m}} \right].$$

a) Show that σ is a well-defined algebraic morphism — it is continuous in the Zariski topology and takes regular functions to regular functions.

b) Find equations for the image of the Segre embedding.

c) Prove that $\mathbb{P}^n \times \mathbb{P}^m$ is isomorphic to the image of the Segre embedding. In particular, show that the Segre embedding is bijective onto the image and the two have the same Zariski topology and the same notion of regular functions.

Solution

a) Let the homogeneous coordinates on $\mathbb{P}^{(n+1)(m+1)-1}$ be given by z_{ij} with $i = 0, \ldots, n, j = 0, \ldots, m$. Then for $F \in \mathbb{C}[z_{ij} : i, j] = \mathbb{C}[\mathbf{z}]$ homogeneous of degree d, we have

$$\sigma^{-1}(V(F)) = V(F(x_0y_0, x_0y_1, \ldots, x_0y_m, x_1y_0, \ldots, x_ny_m))$$

is the vanishing set of a bihomogeneous polynomial of bidegree (d, d). Thus σ is continuous.
Note that the preimage of $U_{ij} = \mathbb{P}^{(n+1)(m+1)-1} \setminus V(z_{ij})$ is given by $U_i \times U_j$ and the restriction of σ to this preimage is

$$\sigma|_{U_i \times U_j} : U_i \times U_j \to U_{ij} \cong \mathbb{C}^{nm}$$
$$([x_0, \ldots, x_n], [y_0, \ldots, y_m]) \mapsto \left(\frac{x_0y_0}{x_iy_j}, \ldots, \frac{x_ny_m}{x_iy_j} \right).$$

Hence all coordinate functions are given as the quotients of the polynomials x_ry_s and x_iy_j, both of bidegree $(1, 1)$. Thus indeed σ is algebraic.

b) We claim that for $i, k \in \{0, \ldots, n\}$ and $j, l \in \{0, \ldots, m\}$, the equations $z_{ij}z_{kl} - z_{il}z_{kj}$ cut out the image of σ. It is immediate that they are homogeneous equations of degree 2, which vanish on the image of σ, as $x_iy_jx_ky_l - x_iy_lx_ky_j = 0$.
On the other hand, assume that $z = [(z_{ij})_{i,j}]$ is in the vanishing locus of all the equations. One of the coordinates z_{ij} must be nonzero and for convenience we say that $z_{00} \neq 0$. By rescaling the numbers z_{ij} we may assume $z_{00} = 1$. Then the equation above for $k = 0, l = 0$ tells us $z_{ij} = z_{i0}z_{0j}$ for all i, j. Thus

$$z = \sigma([1, z_{10}, z_{20}, \ldots, z_{n0}], [1, z_{01}, \ldots, z_{0m}])$$

is indeed contained in the image of σ.

c) In part b) we have actually constructed an inverse σ^{-1} to σ on the open subset $\text{Image}(\sigma) \cap U_{z_{00}}$ and by similar formulas, one defines the inverse map on $\text{Image}(\sigma) \cap U_{z_{ij}}$ for general i, j. Together, this shows already that σ is bijective.
To conclude that σ is an isomorphism, we must show that σ^{-1} is an algebraic morphism. But again this can be checked on the open cover $U_i \times U_j$ of $\mathbb{P}^n \times \mathbb{P}^m$ on the one hand and $\text{Image}(\sigma) \cap U_{z_{ij}}$ on the other hand. For notational convenience, we again consider the case $i = j = 0$.
For a bihomogeneous polynomial $F \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$, its zero set on $U_0 \times U_0$ has as preimage under σ^{-1} the vanishing set

$$V(F(1, z_{10}, z_{20}, \ldots, z_{n0}, 1, z_{01}, \ldots, z_{0m})) \cap \text{Image}(\sigma) \subset U_{z_{ij}} \cong \mathbb{C}^{nm},$$

which is closed in $\mathbb{C}^{nm}$ as the zero locus of a polynomial. Similarly, for an algebraic function F/G on (some subset of) $U_i \times U_j$, the composition $F \circ \sigma^{-1}/G \circ \sigma^{-1}$ is the quotient of two polynomials and hence algebraic.

3. For $d, n \geq 1$, the *d-th Veronese embedding* of $\mathbb{P}^n$ is the map

$$\nu_d : \mathbb{P}^n \longrightarrow \mathbb{P}^{\binom{n+d}{n}-1}$$

whose homogeneous coordinates are given by the $\binom{n+d}{n}$ monomials of degree d in the coordinates $x_0, \ldots, x_n$ of $\mathbb{P}^n$. For instance if $n = 2, d = 2$, we have

$$\nu_2 : \mathbb{P}^2 \to \mathbb{P}^5, [x_0, x_1, x_2] \mapsto [x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2].$$

Prove that ν_d is an algebraic map. Show that the image of ν_d is defined by quadratic equations (find the equations).

Solution As we have seen on the last exercise sheet, a function between projective spaces, whose coordinates are given by homogeneous polynomials of the same degree (that don't vanish simultaneously) is well-defined and algebraic. But indeed the coordinates of ν_d cannot vanish simultaneously, because then in particular $x_0^d = x_1^d = \ldots = x_n^d = 0$, so all $x_i = 0$, a contradiction.
We label the coordinates on $\mathbb{P}^{\binom{n+d}{n}-1}$ by z_α for $\alpha \in (\mathbb{Z}_{\geq 0})^{n+1}$ with $|\alpha| = \alpha_0 + \ldots + \alpha_n = d$. Then the z_α-coordinate of ν_d is exactly $\mathbf{x}^\alpha$. We claim that the image of ν_d is cut out by the quadratic equations

$$z_\alpha z_\beta - z_\gamma z_\delta \text{ for } \alpha + \beta = \gamma + \delta \in (\mathbb{Z}_{\geq 0})^{n+1}. \tag{1}$$

Using $\mathbf{x}^\alpha \cdot \mathbf{x}^\beta = \mathbf{x}^{\alpha+\beta}$, one sees that all the equations above vanish on the image of ν_d.
Assume on the other hand that we have $[z] = [z_\gamma : |\gamma| = d]$ satisfying the equations above. Assume that $z_\alpha \neq 0$ (so we will rescale z such that $z_\alpha = 1$) and let $i \in \{0, \ldots, n\}$ with $\alpha_i > 0$. Define a vector $\mathbf{x} \in \mathbb{C}^{n+1}$ by $x_j = z_{\alpha - e_i + e_j}$ for $j = 0, \ldots, n$. Here $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the i-th unit vector in $(\mathbb{Z}_{\geq 0})^{n+1}$. We claim that there exists a constant $\lambda \in \mathbb{C}$ such that

$$z_\beta = \lambda \mathbf{x}^\beta \text{ for all } \beta \text{ with } |\beta| = d. \tag{2}$$

Showing this finishes the proof, as then firstly $\lambda \neq 0$ as $z \neq 0$ and thus $[z] = [\lambda z] = \sigma([x_0, \ldots, x_n])$. As $x_i = z_\alpha = 1$, we (must) set $\lambda = z_{de_i}$ and then the equation (2) is satisfied for $\beta = de_i$.
We now show that if it is satisfied for some β with $\beta_j > 0$, it is also satisfied for $\beta - e_j + e_k$ for all k. As we can reach every β starting from de_i by sucessively adding vectors $-e_i + e_k$, the statement then holds for all β as claimed. But from equation (1) we get

$$z_\beta z_{\alpha - e_i + e_k} = z_{\beta - e_j + e_k} z_{\alpha - e_i + e_j}.$$

But using the assumption and the definition of $\mathbf{x}$, we get

$$\lambda \mathbf{x}^\beta x_k = z_{\beta - e_j + e_k} x_j$$

and hence $z_{\beta - e_j + e_k} = \lambda \mathbf{x}^{\beta + e_k - e_j}$, which finishes the proof.

4. A *rational normal curve* in $\mathbb{P}^n$ is the composition of $\nu_n : \mathbb{P}^1 \to \mathbb{P}^n$ with a projectively linear transformation, i.e. a map of the form $p \mapsto Ap$ for $A \in \mathrm{GL}_{n+1}$. Let $p_0, ..., p_{n+2}$ be $n+3$ points in general linear position in $\mathbb{P}^n$. Prove that there exists a rational normal curve which passes through the points p_i.

Solution As the points are in general linear position, we can change coordinates on $\mathbb{P}^n$ by a projectively linear transformation, such that for $0 \le i \le n$ we have $p_i = [0, \ldots, 0, 1, 0, \ldots, 0]$, where the 1 is at the i-th position. Consider now $n+1$ (pairwise distinct) points $[a_i, b_i]$ in $\mathbb{P}^1$, $0 \le i \le n$ for some $a_i, b_i \in \mathbb{C}$ with $a_i, b_i \neq 0$. We use homogeneous coordinates x, y on $\mathbb{P}^1$. Let us consider the linear polynomials $\lambda_i = b_i x - a_i y$ and define a function $f : \mathbb{P}^1 \longrightarrow \mathbb{P}^n$ of degree n by

$$f([x,y]) = \left[\prod_{j=1}^{n} \lambda_j(x,y), \ldots, \prod_{j\neq i}^{n} \lambda_j, \ldots, \prod_{j=0}^{n-1} \lambda_j\right]$$

As $\lambda_j(a_i, b_i) = 0$ for $i = j$ and $\neq 0$ otherwise, we have $f([a_i, b_i]) = p_i$ for $i = 0, \ldots, n$. Furthermore, we have

$$f([0,1]) = \left[\left(\prod_{j\neq i}(-a_j)\right)_{i=0,\ldots,n}\right] = \left[\frac{1}{a_0}, \frac{1}{a_1}, \ldots, \frac{1}{a_n}\right], \tag{3}$$

since $\prod_{j\neq i}(-a_j) = (-1)^{n-1} a_0 \cdots a_n / a_i$ and all the a_i are non-zero. Let $p_{n+1} = [c_0, \ldots, c_n]$. As p_{n+1} is in general linear position, $c_i \neq 0$ for all i (otherwise it would be in the span of n of the points). By setting, $a_i = 1/c_i$ we have $f([0,1]) = p_{n+1}$. With the same argument for $[1,0]$ and the b_i, we are done.

5*. Show that for $V \subset \mathbb{C}^m$, $W \subset \mathbb{C}^n$ affine varieties, the set $V \times W \subset \mathbb{C}^{m+n}$ is an affine variety. Compute $I(V \times W)$ and show that the coordinate ring $\Gamma(V \times W)$ is given by $\Gamma(V) \otimes_{\mathbb{C}} \Gamma(W)$.

Solution It is immediately clear that for $I = I(V), J = I(W)$, the ideal (I, J) in $\mathbb{C}[x_1, \ldots, x_m, y_1, \ldots, y_n]$ generated by the ideals of V, W has exactly the vanishing locus $V \times W$. Here we use the inclusion $I \subset \mathbb{C}[x_1, \ldots, x_m] \subset \mathbb{C}[x_1, \ldots, x_m, y_1, \ldots, y_n]$ (similar for J). We claim that $I(V \times W) = (I, J)$. Once we have this, the coordinate ring of $V \times W$ is exactly

$$\begin{aligned}\Gamma(V \times W) &= \mathbb{C}[x_1, \ldots, x_m, y_1, \ldots, y_n]/(I, J) \\ &= \mathbb{C}[x_1, \ldots, x_m]/I \otimes_{\mathbb{C}} \mathbb{C}[y_1, \ldots, y_n]/J = \Gamma(V) \otimes_{\mathbb{C}} \Gamma(W).\end{aligned}$$

The proof of $I(V \times W) = (I, J)$ uses that we work over the algebraically closed field $\mathbb{C}$. A very good account of such a proof can be found in the following math-stackexchange thread:

http://www.math.stackexchange.com/questions/731218/generators-for-radical-ideal-of-product-of-affine-varieties.

Due April 4.