Algebraic Geometry

D-MATH, FS 2016

Prof. Pandharipande

Exercise Sheet 5 - Solutions

1. Find the equations defining the image of the following algebraic map.

$$f : \mathbb{P}^1 \to \mathbb{P}^2$$
$$[x, y] \longmapsto [x^3, x^2y, y^3]$$

Solution Denote the coordinates on $\mathbb{P}^2$ by $[z_0, z_1, z_2]$ and let $h = z_1^3 - z_0^2 z_2$.

Claim: $\mathrm{im}(f) = V(h)$.

This can be seen as follows. On $U = \{z_0 \neq 0\}$, define the inverse map

$$g : U \cap V(h) \to \mathbb{P}^1, \quad [z_0, z_1, z_2] \mapsto [1, z_1/z_0]$$

Its easy to check that $(g \circ f)|_{\{x \neq 0\}} = id$ and $f \circ g = id_{U \cap V(h)}$. This shows that $U \cap V(h) \cong \{x \neq 0\} \cong \mathbb{C} \subset \mathbb{P}^1$.

On the other hand, if $P = [z_0, z_1, z_2] \in V(h)$ with $z_0 = 0$, then $z_1 = 0$ so that $P = f([0, 1])$. This shows that f is a bijection of $\mathbb{P}^1$ onto $V(h)$ (but not a biregular map as $V(h)$ is singular) and hence we are done.

2. Let $f = \nu_n : \mathbb{P}^1 \to \mathbb{P}^n, [x, y] \mapsto [x^n, x^{n-1}y, \dots, y^n]$ be the Veronese map. Let $p_1, \dots, p_{n+1}$ be *distinct* points in $\mathbb{P}^1$. Prove that $f(p_1), \dots, f(p_{n+1})$ are in general linear position in $\mathbb{P}^n$.

Solution Set $f = \nu_n$ the Veronese embedding. The condition that the points $f(p_1), \dots, f(p_{n+1})$ are in general linear position means that they do not lie on a hyperplane. So assume to the contrary that $f(p_1), \dots, f(p_{n+1}) \in V(\lambda)$ for λ a non-zero linear homogeneous polynomial. Then $\lambda(f(x, y))$ is a polynomial of degree n, with $n + 1$ distinct roots. Hence $\lambda(f(x, y)) = 0$, and then so is λ. A contradiction.

3. Let $X \subset \mathbb{P}^n$ be a projective variety which is not a finite collection of points. Let $G_d \in \mathbb{C}[x_0, \dots, x_n]$ be a homogeneous polynomial of degree $d > 0$. Prove that the zero locus of G_d must intersect X. (Hint: If $V(G_d)$ is disjoint from X, use this to define a non-constant function on some connected component.)

Solution There are two ways to prove this statement. Either use the Hint to construct global non-constant functions f/G_d for f any homogeneous polynomial of degree d. Otherwise, consider the d-th Veronese embedding $\nu_d : \mathbb{P}^n \to \mathbb{P}^N$ with image Y. There is a linear polynomial λ on $\mathbb{P}^N$ such that $Y \cap V(\lambda) = \nu_d(V(G_d))$. If $V(G_d)$ is disjoint from X, then X is a projective variety contained in $\mathbb{P}^N \setminus V(\lambda) \cong \mathbb{C}^N$. But any projective variety that is affine is a finite collection of points.

Here we use that an algebraic function on a projective variety X is locally constant. Indeed, if we have a function $f : X \to \mathbb{C} \subset \mathbb{P}^1$, then the image of f is a projective subvariety in $\mathbb{P}^1$, which cannot be all of $\mathbb{P}^1$, as it is contained in $\mathbb{C}$. Thus, the image is a finite collection of points, so f is constant on the connected components of X.

4. Let X be a quasi-projective variety. Prove that X does not admit an infinite chain of strictly decreasing Zariski closed subsets.

Solution Let $X \subset \mathbb{P}^n$ be a quasi-projective variety and let $Z_\bullet = (Z_1 \supset Z_2 \supset Z_3 \supset \dots)$ be an decreasing sequence of closed subsets of X. Let $\overline{X} \subset \mathbb{P}^n$ be the closure of X and denote with $\overline{Z}_\bullet = (\overline{Z_1} \supset \overline{Z_2} \supset \dots)$ the closure of the sequence in $\mathbb{P}^n$. If $x_0, \dots, x_n$ are the coordinates, then $D(x_i) = \{x_i \neq 0\} \cap \overline{X}$ is an affine variety and therefore $\overline{Z}_\bullet \cap \{x_i \neq 0\}$ stabilizes (by the correspondence between closed subsets and radicals ideals of $\mathbb{C}[z_1, \dots, z_n]$ and the fact that $\mathbb{C}[z_1, \dots, z_n]$ is Noetherian). As $D(x_i), i = 0, \dots, n$ cover $\mathbb{P}^n$, $\overline{Z}_\bullet$ must stabilize and hence so must $Z_\bullet = \overline{Z}_\bullet \cap X$.

5. For every positive integer n, let L_n be the line in $\mathbb{P}^3$ defined by

$$\left\{ [x, y, nx, ny] \;\middle|\; [x, y] \in \mathbb{P}^1 \right\}$$

Find the equations defining the Zariski closure of the infinite union

$$L_1 \cup L_2 \cup L_3 \cup \dots$$

Solution Let x_0, x_1, x_2, x_3 be the coordinates on $\mathbb{P}^3$. We obviously have that $\cup_i L_i \subset X = V(x_0x_3 - x_1x_2)$, and hence $\overline{\cup_i L_i} \subset X$. Therefore it is enough to consider the problem inside X. But we have seen before that X is the image of the Segre embedding

$$\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3, \quad ([a_0, a_1], [b_0, b_1]) \mapsto [a_0b_0, a_0b_1, a_1b_0, a_1b_1]$$

Under σ, the lines L_n correspond to $[1/n, 1] \times \mathbb{P}^1 \subset \mathbb{P}^1 \times \mathbb{P}^1$ and we have reduced to find the closure of $A = \bigcup_n [1/n, 1] \times \mathbb{P}^1$ inside $\mathbb{P}^1 \times \mathbb{P}^1$.

Consider the horizontal slice $A \cap (\mathbb{P}^1 \times P)$ for some $P \in \mathbb{P}^1$. The closure of $A \cap (\mathbb{P}^1 \times P)$ inside $\mathbb{P}^1 \times P$ is isomorphic to the closure of $\{[1/n, 1] \mid n \in \mathbb{N}\}$ inside $\mathbb{P}^1$. But the closure of $\{[1/n, 1] \mid n \in \mathbb{N}\}$ inside $\mathbb{P}^1$ is just all of $\mathbb{P}^1$ (as every polynomial that vanishes at infinitely many points is zero). We conclude

$$\overline{A} \supset \bigcup_{P \in \mathbb{P}^1} \overline{A \cap (\mathbb{P}^1 \times P)} = \bigcup_{P \in \mathbb{P}^1} \mathbb{P}^1 \times P = \mathbb{P}^1 \times \mathbb{P}^1$$

and therefore

$$\overline{\bigcup_n L_n} = V(x_0x_3 - x_1x_2)$$

6. Let $X = \mathbb{P}^2 \setminus \{\text{point}\}$. Is there a non-constant algebraic map $f : X \to \mathbb{C}$?

Solution Let $f : X \to \mathbb{C}$ be a global regular function (i.e. an algebraic map to $\mathbb{C}$). Then restricting to a standard covering $U \cong \mathbb{C}^2 \subset \mathbb{P}^2$ that contains the point P, we can see by problem 3 of sheet 2 that this function extends to all of $\mathbb{P}^2$, hence must be constant.

Alternatively, for any regular map $g : \mathbb{P}^1 \to X$, the composition $f \circ g : \mathbb{P}^1 \to \mathbb{C}$ must be constant. We see that f is constant on each line $L \subset X$. But for any two points p, q on X we can find lines L_0, L_1 such that $p \in L_0$, $q \in L_1$ and $L_0 \cap L_1 \neq \varnothing$. Hence f must be constant.

Due April 15.