Exercise Sheet 6 - Solutions

1. Let $n \geq 2$ and let $\omega \in \bigwedge^2 \mathbb{C}^n$ be nonzero.

a) Show that there exists $k \geq 1$ such that

$$\omega = v_1 \wedge v_2 + \ldots + v_{2k-1} \wedge v_{2k}$$

for $v_1, \ldots, v_{2k} \in \mathbb{C}^n$ linearly independent.

b) Prove that

$$\omega^{\wedge k} = \underbrace{\omega \wedge \omega \wedge \ldots \wedge \omega}_{k \text{ times}} \neq 0$$

but $\omega^{\wedge k+1} = 0$. In particular, k is uniquely determined and it is called the *rank* of ω. If ω is of rank 1, it is called *decomposable*.

c) Consider the map

$$s(\omega) : \mathbb{C}^n \to \bigwedge^3 \mathbb{C}^n, v \mapsto v \wedge \omega.$$

Show that it has trivial kernel if the rank of ω is greater than 1 and that the kernel has dimension 2 if the rank is equal to 1.

d) Let $\mathrm{Gr}(2, n)$ be the Grassmannian of 2-planes in $\mathbb{C}^n$ and let

$$\gamma : \mathrm{Gr}(2, n) \to \mathbb{P}(\bigwedge^2 \mathbb{C}^n)$$

be the Plücker embedding given by $\gamma(\Lambda) = [v_1 \wedge v_2]$, where $\Lambda \in \mathrm{Gr}(2, n)$ and $v_1, v_2 \in \mathbb{C}^n$ is a basis of Λ.
Show that γ is a bijection onto the set X of classes of decomposable vectors in $\mathbb{P}(\bigwedge^2 \mathbb{C}^n)$.

e) Let $(e_i)_{i=1,\ldots,n}$ be the standard basis on $\mathbb{C}^n$. Let $(e_i \wedge e_j)_{1 \leq i < j \leq n}$ be the induced basis for $\bigwedge^2 \mathbb{C}^n$. Writing each element $\omega \in \bigwedge^2 \mathbb{C}^n$ as $\omega = \sum_{i<j} a_{ij} e_i \wedge e_j$ induces homogeneous coordinates $[(a_{ij})_{i<j}]$ on $\mathbb{P}(\bigwedge^2 \mathbb{C}^n)$. Use *b)* to show that X is cut out by quadratic equations in the coefficients a_{ij}.
In particular, for $n = 4$ we have that X is cut out by the famous Plücker quadric

$$X = V(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}).$$

Solution

a) Consider the minimal k such that we can write

$$\omega = v_1 \wedge v_2 + \ldots + v_{2k-1} \wedge v_{2k} \tag{1}$$

for some $v_1, \ldots, v_{2k} \in \mathbb{C}^n$. We claim that then the v_i are automatically linearly independent. Indeed, assume we can write

$$v_{2k} = a_1 v_1 + \ldots + a_{2k-1} v_{2k-1}.$$

Then we can set $a_{2k-1} = 0$ without changing the term (1), as $v_{2k-1} \wedge v_{2k-1} = 0$. Moreover, we can expand the term $v_{2k-1} \wedge v_{2k}$ bilinearly and obtain a representation of ω as a sum of $k-1$ wedge products, a contradiction to our assumption. Indeed

$$v_1 \wedge v_2 + v_{2k-1} \wedge (a_1 v_1 + a_2 v_2) = \begin{cases} (v_1 + a_2 v_{2k-1}) \wedge v_2 & \text{if } a_1 = 0, \\ (v_1 + (a_2/a_1) v_2) \wedge (v_2 - a_1 v_{2k-1}) & \text{if } a_1 \neq 0. \end{cases}$$

Similarly we can integrate the other terms $v_{2k-1} \wedge (a_{2i-1} v_{2i-1} + a_{2i} v_{2i})$ into the existing summands of (1), obtaining a contradiction.

b) Let $v_1, \ldots, v_{2k}$ be as in a) and enrich them to a basis $v_1, \ldots, v_n$ of $\mathbb{C}^n$. Then

$$\omega^{\wedge k} = k! v_1 \wedge v_2 \wedge v_3 \wedge \ldots \wedge v_{2k}.$$

As $v_1 \wedge v_2 \wedge v_3 \wedge \ldots \wedge v_{2k}$ is an element of the canonical basis of $\bigwedge^{2k} \mathbb{C}^n$ induced by $v_1, \ldots, v_n$, the vector $\omega^{\wedge k}$ is nonzero. On the other hand from the above formula it is obvious that $\omega^{\wedge k+1} = \omega \wedge \omega^{\wedge k} = 0$.

c) With notation as in the previous exercise part let $v = b_1 v_1 + \ldots + b_n v_n \in \mathbb{C}^n$ be any vector. Then we have that in $s(\omega)(v)$, the coefficient of $v_1 \wedge v_2 \wedge v_i$ is b_i for $3 \leq i \leq n$. Thus if $s(\omega)(v) = 0$, we have $v = b_1 v_1 + b_2 v_2$. For $k = 1$, i.e. $\omega = v_1 \wedge v_2$ all v of this form are indeed in the kernel. For $k > 1$ we see for instance that b_i is the coefficient of $v_i \wedge v_3 \wedge v_4$ in $s(\omega)(b_1 v_1 + b_2 v_2)$ for $i = 1, 2$, so $s(\omega)$ is injective as claimed.

d) By definition, the image of γ is exactly the locus X of decomposable vectors. To show that γ is injective, note that for $\omega = \gamma(\mathrm{Span}(v_1, v_2))$ we have

$$\mathrm{Ker}(s(\omega)) = \mathrm{Ker}(s(v_1 \wedge v_2)) = \mathrm{Span}(v_1, v_2),$$

so we can reconstruct $\mathrm{Span}(v_1, v_2) \in \mathrm{Gr}(2, n)$ from its image under γ.

e) By part b) we know that $[\omega] \in X$ if and only if $\omega \wedge \omega = 0$. Writing this equation for $\omega = \sum_{i<j} a_{ij} e_i \wedge e_j$, we obtain

$$\omega \wedge \omega = \sum_{1 \leq i < j < k < l \leq n} (a_{ij} a_{kl} - a_{ik} a_{jl} + a_{il} a_{jk}) e_i \wedge e_j \wedge e_k \wedge e_l.$$

Thus X is cut out by the quadrics $a_{ij} a_{kl} - a_{ik} a_{jl} + a_{il} a_{jk}$.

2. Let $L \subset \mathbb{P}^3$ be a fixed line. Let $H_L \subset \mathrm{Gr}(2, 4)$ be the locus of lines which meet L.

a) Consider the incidence subset

$$I = \{(p, l) \in \mathbb{P}^3 \times \mathrm{Gr}(2, 4) : p \in l\} \subset \mathbb{P}^3 \times \mathrm{Gr}(2, 4).$$

Show that I is a closed subvariety of $\mathbb{P}^3 \times \mathrm{Gr}(2, 4)$.

b) Conclude that that H_L is a closed subvariety of $\mathrm{Gr}(2, 4)$.

c) What is the image of H_L under the Plücker embedding in $\mathbb{P}^5$?

d) Use this geometry to give a different proof of the fact that there are 2 lines meeting 4 fixed general lines in space.

Solution

a) We can check that I is closed on some open cover of $\mathbb{P}^3 \times \mathrm{Gr}(2, 4)$. Recall that $\mathrm{Gr}(2, 4)$ has the open cover $U_{i,j}$ for $1 \leq i < j \leq 4$, where

$$\mathbb{C}^4 \cong U_{1,2} = \{\mathrm{Span}\left((1, 0, a, c), (0, 1, b, d)\right)\} \subset \mathrm{Gr}(2, 4).$$

The other $U_{i,j}$ are defined similarly and we will only present the proof that $I \cap \mathbb{P}^3 \times U_{1,2}$ is closed in $\mathbb{P}^3 \times U_{1,2}$, the other proofs are similar. Now an element $p = [w, x, y, z] \in \mathbb{P}^3$ is contained in the line l corresponding to $(a, b, c, d) \in \mathbb{C}^4 \cong U_{1,2}$ if and only if we can write

$$(w, x, y, z) = \lambda(1, 0, a, c) + \mu(0, 1, b, d)$$

for $\lambda, \mu \in \mathbb{C}$. But comparing first and second coordinate of both sides, we see $\lambda = w, \mu = x$, so

$$\begin{aligned} &I \cap \mathbb{P}^3 \times U_{1,2} \\ =&\{([w, x, y, z], (a, b, c, d)) : y = wa + xb, z = wc + xd\} \subset \mathbb{P}^3 \times U_{1,2}. \end{aligned}$$

The equations $y = wa + xb, z = wc + xd$ are homogeneous polynomials in w, x, y, z, whose coefficients are polynomials in a, b, c, d. So indeed the subset above is closed.

b) Consider the incidence variety $I \subset \mathbb{P}^3 \times \mathrm{Gr}(2, 4)$ and denote with π_1, π_2 the projection on the both factors. We have

$$H_L = \pi_2(\pi_1^{-1}(L) \cap I)$$

This shows that H_L is closed: $\pi_1^{-1}(L)$ is closed, since π_1 is continous. $\pi_1^{-1}(L) \cap I$ is closed since I is closed. $\pi_2(\pi_1^{-1}(L) \cap I)$ is closed, since the image of a projective variety under an algebraic map is closed.

c) Let $\hat{L} = \mathrm{Span}(v_1, v_2) \subset \mathbb{C}^4$ be such that $L = \mathbb{P}(\hat{L})$ for some linearly independent v_1, v_2. Define $\varphi_L : \bigwedge^2(\mathbb{C}^4) \to \bigwedge^4(\mathbb{C}^4) \cong \mathbb{C}$ by $\varphi_L(\omega) = \omega \wedge (v_1 \wedge v_2)$. This defines a homogeneous linear form on $\mathbb{P}(\bigwedge^2(\mathbb{C}^4))$. Moreover, if $\omega = w_1 \wedge w_2$, then $\varphi_L(\omega) = 0$ iff $aw_1 + bw_2 \in \mathrm{Span}(v_1, v_2)$ for some $(a, b) \neq 0$. This shows that the image of H_L under the Plücker embedding is given by $V(\varphi_L) \cap X$, where X is the image of $\mathrm{Gr}(2, 4)$ inside $\mathbb{P}^5$.

d) Pick $L_1, \dots, L_4$ in general position. The statement we want to prove is exactly that the intersection of the sets H_{L_i} are two points. From c) we know that $H_{L_i} = V(\varphi_L) \cap X$, so we reduce to showing that $X \cap V(\varphi_{L_1}) \cap \dots \cap V(\varphi_{L_4})$ are two points.

Claim. The associated hyperplanes $V(\varphi_{L_1}), \dots, V(\varphi_{L_4})$ are in general linear position and therefore their intersection $V(\varphi_{L_1}) \cap \dots \cap V(\varphi_{L_4})$ is a line $L' \subset \mathbb{P}^5$.

Assuming the claim, we can conclude easily: Since the line L' meets the quadric X in precisely 2 points, we are done.

Proof of Claim. The pairing

$$\bigwedge^2 \mathbb{C}^4 \otimes \bigwedge^2 \mathbb{C}^4 \to \bigwedge^4 \mathbb{C}^4 \cong \mathbb{C}, \quad \alpha \otimes \beta \mapsto \alpha \wedge \beta$$

is non-degenerate and hence induces an isomorphism $\bigwedge^2(\mathbb{C}^4) \to \left(\bigwedge^2 \mathbb{C}^4\right)^*$. Taking the projectivisation, we obtain

$$F : \mathbb{P}(\bigwedge^2(\mathbb{C}^4)) \cong \mathbb{P}\left(\bigwedge^2 \mathbb{C}^4\right)^*$$

Under F, the set $\{[\varphi_L] \mid L \in \mathrm{Gr}(2,4)\}$ on the right corresponds precisely to the quadric $X = \mathrm{Gr}(2,4)$ on the left. Hence picking 4 general lines is picking 4 general points on this quadric is picking 4 general hyperplanes, and the claim follows.

3. Let $Q_2 \subset \mathbb{P}^3$ be the quadric given by the equation $x_0x_3 - x_1x_2$. Find (with proof) all lines which lie on Q_2.

Solution

A line in $\mathbb{P}^3$ is the image of a degree 1 map $a : \mathbb{P}^1 \to \mathbb{P}^3$, i.e. given by polynomials of degree 1. We have seen that the Segre map $\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \to X = V(x_0x_3 - x_1x_2) \subset \mathbb{P}^3$ is an isomorphism of algebraic varieties. Hence every map $f : \mathbb{P}^1 \to \mathbb{P}^3$ with image inside X factors as $\sigma \circ \hat{f}$ for a map $\hat{f} : \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$. Let $\hat{f}_1, \hat{f}_2$ be the composition of $\hat{f}$ with projection of $\mathbb{P}^1 \times \mathbb{P}^1$ to the respective factors. Let d_i be the degree of $\hat{f}_i$, $i = 1, 2$. Then from the definition of the Segree map we see that the degree of f is given by $d_1 + d_2$.

So if $a : \mathbb{P}^1 \to \mathbb{P}^3$ is a degree 1 map with image in X, either $d_1 = 0$ (and $\hat{f}_1$ constant) or $d_2 = 0$. So every line contained in X is given by the image of $p \times \mathbb{P}^1$ or $\mathbb{P}^1 \times q$ for $p, q \in \mathbb{P}^1$ under the Segre map σ.

4. Let $X \subset \mathbb{P}(\bigwedge^2 \mathbb{C}^4) = \mathbb{P}^5$ be the quadric which is the image of the Plücker embedding of $\mathrm{Gr}(2,4)$. Find all planes $\mathbb{P}^2 \subset \mathbb{P}^5$ (linearly embedded) which lie on X. Can you interpret these in terms of the geometry of $\mathrm{Gr}(2,4)$?

Solution

Let $\mathbb{P}^2 \cong T \subset X \subset \mathbb{P}^5$ be linearly embedded and pick a point $x = [\hat{x}] \in T$ with $\hat{x} \in \bigwedge^2 \mathbb{C}^4 \setminus \{0\}$. For any other point $y = [\hat{y}] \in T$, we have $(\hat{x} + t\hat{y}) \wedge (\hat{x} + t\hat{y}) = 0$ where $t \in \mathbb{C}$. Differentiating by t, we get that $\hat{x} \wedge \hat{y} = 0$. As for φ_L in problem 2, this means that the lines $L_x, L_y \subset \mathbb{P}^3$, which x and y represent, meet at a point P_{xy}. Let now $x, y, z \in T$ be points spanning T and consider the three lines L_x, L_y, L_z plus their points of intersection P_{xy}, P_{xz}, P_{yz}. By the same argument, we have that for any other point $w \in T$, the associated line L_w meets L_x, L_y, L_z.

(i) If two of the points P_{xy}, P_{xz}, P_{yz} coincide in a point P, then so do all three and the corresponding lines span all of $\mathbb{P}^3$. If $w \in T$ is some other point, then the line L_w must meet these three lines, hence must go through P. Conversely any line L that goes through P meets L_x, L_y, L_z and hence $L \in V(\varphi_{L_x}) \cap V(\varphi_{L_y}) \cap V(\varphi_{L_z}) = T$. This shows

$$T = \{ \text{ lines meeting a fixed point } P\}$$

(ii) If all three points are different, then they span a plane $W \cong \mathbb{P}^2 \subset \mathbb{P}^3$ and $L_x, L_y, L_z \subset W$. For any other line L_w we have that the condition meeting these three lines implies that it is contained in W. As the converse is true as well, we have in this case

$$T = \{ \text{ lines lying in a fixed 2-plane } W \subset \mathbb{P}^3\}$$

This gives two 3-dimensional families of linear planes inside X, parametrized by $P \in \mathbb{P}^3$ and $\lambda \in (\mathbb{P}^3)^*$ respectively.

5. a) Let $X, Y \subset \mathbb{C}^n$ be subvarieties and assume Y is cut out by polynomials $f_1, \dots, f_r \in \mathbb{C}[x_1, \dots, x_n]$. Then for $p \in X \cap Y$ show that $\dim T_p X \cap Y \geq \dim T_p X - r$.

b) Consider $X = V(xy - z^2) \subset \mathbb{C}^3$. Find irreducible curves $C_1, C_2, C_3 \subset X$ going through the origin and satisfying $\dim T_0 C_i = i$ for $i = 1, 2, 3$.

c) Show that the line $l \subset X$ spanned by $(0, 1, 0)$ is not cut out from X by a single polynomial.

Solution

a) The tangent space $T_p(X \cap Y)$ to $X \cap Y$ at p is cut out from the vector space $T_p X$ by the linear equations $df_1|_p, \dots, df_r|_p$. Thus its dimension must at least be $\dim T_p X - r$.

b) We can define the curves C_i as

C_1: $t \mapsto (t, 0, 0)$

C_2: $t \mapsto (t^2, t^4, t^3)$ cut out by $xy = z^2, x^2 - y = 0$

C_3: $t \mapsto (t^3, t^7, t^5)$ cut out by $y^3 = x^7, z^3 = x^5, y^5 = z^7$.

Let $P = (0, 0, 0)$. $\dim T_P(C_1) = 1$ is clear. For C_2, C_3, note that C_2 is the intersection of the smooth surface $S' = V(x^2 - y)$ with the surface S that has a singularity at P (and therefore $\dim T_P S = \dim \mathbb{C}^3 = 3$). Hence $\dim T_P C_2 = \dim T_P S' \cap T_P S = \dim T_P S' = 2$. As C_3 is the intersection of three surfaces, each having a singularity at P, we conclude likewise $\dim T_P C_3 = 3$.

The parametrization given above shows additionally that the curves are indeed irreducible.

c) It is clear that $\dim T_0 l = 1$ and we have seen $\dim T_0 X = 3$. Hence by a) we have $\dim T_0(X \cap V(g)) \geq 2$ for all polynomials g with $g(0) = 0$. Thus l cannot be cut out by a single such polynomial.

Due April 22.