Exercise Sheet 7 - Solutions

1. Prove that the Zariski tangent space at the point $[S] \in \mathrm{Gr}(r, V)$ is canonically isomorphic to $S^* \otimes V/S$ (or equivalently to $\mathrm{Hom}(S, V/S)$).

Solution See e.g. *Harris, Algebraic Geometry*, Example 16.1

2. Let $M_{n,n}$ be the space of $n \times n$ complex matrices. Let $\det : M_{n,n} \to \mathbb{C}$ be the determinant and let $Z = V(\det)$ be its zero set. Find all the points of Z such that the Zariski tangent space to Z has dimension exactly $n^2 - 1$.

Solution Let $A \in M_{n,n}$ be a matrix and let X_{kl} be coordinates on $M_{n,n}$. We want to compute the derivative of det with respect to the variable X_{kl} at the point A. For this let $E_{kl} \in M_{n,n}$ be the matrix with coefficient 1 at the (k,l)-th position and 0 otherwise. By expanding $\det(A + tE_{kl})$ in the k-th column, we have

$$\begin{aligned} \frac{\partial \det}{\partial X_{kl}} &= d(\det)_A(E_{kl}) = \frac{d}{dt}\Big|_{t=0} \det(A + tE_{kl}) \\ &= \frac{d}{dt}\Big|_{t=0} \sum_{j=1}^{n} (-1)^{k+j}(a_{kj} + \delta_{jl}t)M_{kj} \\ &= (-1)^{k+l} M_{kl} \end{aligned}$$

where M_{ij} is the (i,j)-th minor of A. In general we know that for $A \in Z$ the tangent space $T_A Z$ is cut out from $T_A M_{n,n} = \mathbb{C}^{n^2}$ by $d(\det)$, so it is of dimension at least $n^2 - 1$ with equality iff $d(\det) \neq 0$. Hence for $A \in V(\det)$,

$$\begin{aligned} & d(\det) = 0 \\ \iff \quad & \text{all partial derivatives of } \det \text{ vanish at } A \\ \iff \quad & M_{kl} = 0 \text{ for all } k, l \\ \iff \quad & \mathrm{rank}(A) \leq n - 2 \end{aligned}$$

Hence the locus where the Zariski tangent space of $V(\det)$ has dimension $n^2 - 1$ are the matrices of rank exactly $n - 1$.

3. In this exercise, we want to show the following result about intersections of subvarieties of $\mathbb{P}^n$.

Lemma 1. *Let $X, Y \subset \mathbb{P}^n$ be subvarieties such that $\dim(X) + \dim(Y) \geq n$. Then $X \cap Y \neq \emptyset$.*

Below, we will first prove this result in the case that Y is a linear subspace of $\mathbb{P}^n$ and then see how to show the general case from this.

a) For $X \subset \mathbb{P}^n$ a subvariety let

$$\hat{X} = \{p \in \mathbb{C}^{n+1} \setminus \{0\} : [p] \in X\} \cup \{0\} \subset \mathbb{C}^{n+1}$$

be the *affine cone* over X. Show that $\hat{X}$ is an affine variety and that it is irreducible if X is irreducible. *Hint:* For irreducibility, given open sets V_1, V_2 in $\hat{X}$, construct open sets in X by intersecting with an affine hyperplane in $\mathbb{C}^{n+1}$.

b) Prove that $\dim(\hat{X}) = \dim(X) + 1$. *Hint:* Look at the intersections of $\hat{X}$ with $\hat{U}_i = \{(x_0, \ldots, x_n) : x_i \neq 0\}$.

c) Let $L \subset \mathbb{P}^n$ be a linear subspace of codimension c (i.e. dimension $n - c$). Show that $\dim(X \cap L) \geq \dim(X) - c$. In particular, if $\dim(X) \geq c$ we have $X \cap L \neq \emptyset$. You may use the following result, a geometric version of Krull's principal ideal theorem.

Theorem 1. *If X is an affine variety and f is a regular function on X, then* $\dim(V(f)) \geq \dim(X) - 1$.

d) Let $X, Y \subset \mathbb{P}^n$ be subvarieties. Consider $\mathbb{P}^{2n+1}$ with coordinates $x_0, \ldots, x_n$, $y_0, \ldots, y_n$ and identify $X \subset \mathbb{P}^n = V(y_0, \ldots, y_n) \subset \mathbb{P}^{2n+1}$ and similarly $Y \subset \mathbb{P}^n = V(x_0, \ldots, x_n) \subset \mathbb{P}^{2n+1}$. Consider the *join* $J(X, Y)$ of X and Y defined by

$$J(X, Y) = \{[t(x, 0) + s(0, y)] : [x] \in X, [y] \in Y, (s, t) \in \mathbb{C} \setminus \{0\}\} \subset \mathbb{P}^{2n+1}.$$

Show that $J(X, Y)$ is a subvariety of dimension $\dim(X) + \dim(Y) + 1$. Here you can use the following result, which will be proved later.

Theorem 2. *If X, Y are irreducible quasi-projective varieties, we have* $\dim(X \times Y) = \dim(X) + \dim(Y)$.

e) Identify $X \cap Y$ as an intersection of $J(X, Y)$ with $n + 1$ hyperplanes in $\mathbb{P}^{2n+1}$. Conclude that it is nonempty if $\dim(X) + \dim(Y) \geq n$.

f) Give an example of a projective variety Z with closed subvarieties X, Y satisfying $\dim(X) + \dim(Y) \geq \dim(Z)$ but $X \cap Y = \emptyset$.

Solution

a) If $F_1, \ldots, F_r \in \mathbb{C}[x_0, \ldots, x_n]$ are homogeneous polynomials such that $X = V(F_1, \ldots, F_r)$, then $\hat{X} = V(F_1, \ldots, F_r) \subset \mathbb{C}^{n+1}$ and so indeed $\hat{X}$ is an algebraic variety.
In the case that X is irreducible and let $V_1, V_2 \subset \hat{X}$ be nonempty open subsets. We want to show that they intersect.

Assume otherwise, then as for every $p \in \mathbb{C}^{n+1}$ the line $\mathbb{C}p \subset \mathbb{C}^{n+1}$ is irreducible, we must have $(\mathbb{C}p) \cap V_1 = \emptyset$ or $(\mathbb{C}p) \cap V_2 = \emptyset$. From this we see that we can choose $p_1 \in V_1, p_2 \in V_2$ not on a line through the origin. Then there exists an affine linear hyperplane $H \subset \mathbb{C}^{n+1}$ containing p_1, p_2

and not containing 0. The quotient map $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ restricted to H defines an isomorphism of H with an open subset of $\mathbb{P}^n$. A special case of this are the usual isomorphisms

$$H_i = \{(x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots x_n)\} \to U_i \subset \mathbb{P}^n.$$

Intersecting with $\hat{X}$, we see that $\hat{X} \cap H$ is isomorphic to an open subset of X. The open subsets V_i thus induce open subsets $V_i \cap \hat{X} \cap H$ in X and these are nonempty because $p_i \in V_i \cap \hat{X} \cap H$. But we assumed that the V_i do not intersect, giving a contradiction to the irreducibility of X.

b) To show the equality concerning the dimensions, we can consider all irreducible components of X separately and therefore restrict to the case where X is irreducible. Here we know by the previous exercise part that then also $\hat{X}$ is irreducible. To compute dimensions of $X, \hat{X}$, we will use that $\dim(Y) = \mathrm{trdeg}_{\mathbb{C}} K(Y)$ for Y an irreducible variety. Choose i such that $X \cap U_i \neq \emptyset$, then for $R = \Gamma(X \cap U_i)$ the ring of functions on the affine variety $X \cap U_i$, we have

$$\dim(X) = \dim(X \cap U_i) = \mathrm{trdeg}_{\mathbb{C}} Q(R),$$

where $Q(R)$ is the fraction field of the ring R. On the other hand, for the open set $\hat{U}_i = \{(x_0, \dots, x_n) \in \mathbb{C}^{n+1} : x_i \neq 0\}$ we have

$$\dim(\hat{X}) = \dim(\hat{X} \cap \hat{U}_i) = \mathrm{trdeg}_{\mathbb{C}} Q(\Gamma(\hat{X} \cap \hat{U}_i))$$

for the same reason. But we can identify the variety $\hat{X} \cap \hat{U}_i$: it is isomorphic to $(X \cap U_i) \times \mathbb{C}^*$ via the map

$$\hat{X} \cap \hat{U}_i \to (X \cap U_i) \times \mathbb{C}^*, (x_0, \dots, x_n) \mapsto ([x_0, \dots, x_n], x_i).$$

The inverse of this map is given by

$$([y_0, \dots, y_n], \lambda) \mapsto (\lambda y_0/y_i, \dots, \lambda y_{i-1}/y_i, \lambda, \lambda y_{i+1}/y_i, \dots, \lambda y_n/y_i).$$

But as we have seen on a previous sheet

$$\Gamma((X \cap U_i) \times \mathbb{C}^*) = \Gamma(X \cap U_i) \otimes \Gamma(\mathbb{C}^*) = R \otimes \mathbb{C}[t, t^{-1}] = R[t, t^{-1}].$$

By elementary commutative algebra, we have

$$Q(R[t, t^{-1}]) = Q(R)(t),$$

so indeed $\mathrm{trdeg}_{\mathbb{C}}(Q(R[t, t^{-1}])) = \mathrm{trdeg}_{\mathbb{C}}(Q(R)) + 1$ as desired (if $y_1, \dots, y_d$ are a transcencence basis of $Q(R)$, the elements $y_1, \dots, y_d, t$ are a transcencence basis of $Q(R)(t)$).

c) It is clear that if we show the case of codimension $c = 1$, the general case follows by induction on c, as every linear subspace of codimension c is an intersection of c linear subspaces of codimension 1.
So let $L = V(F)$ be a hyperplane, where F is a linear form in $x_0, \dots, x_n$. It is clear that the affine cone $\widehat{X \cap L}$ of $X \cap L$ is exactly $\hat{X} \cap \hat{L}$, where

$\hat{L} = V(F) \subset \mathbb{C}^{n+1}$ is the affine cone over L, which is still the vanishing set of F. But then by Krull's principal ideal theorem, we have $\dim(\hat{X} \cap V(F)) \geq \dim(\hat{X}) - 1$, so

$$\dim(X \cap L) = \dim(\hat{X} \cap V(F)) - 1 \geq \dim(\hat{X}) - 2 = \dim(X) - 1.$$

Now let L again be any linear subspace of codimension c. To show that $X \cap L$ is nonempty if $\dim(X) \geq c$ simply note that by the above argument, we showed $\widehat{X \cap L}$ has dimension at least 1. Hence it cannot just be the set $\{0\} \subset \mathbb{C}^{n+1}$ (which has dimension 0), but must contain a nonzero vector p. Then $[p] \in X \cap L$.

d) We check that $J(X, Y)$ is a closed subset of $\mathbb{P}^{2n+1}$ on the standard open cover $(U_i)_{i=0,\dots,2n+1}$ of $\mathbb{P}^{2n+1}$. Assume $0 \leq i \leq n$, then we note that $U_i = V_i \times \mathbb{C}^{n+1}$, where $V_0, \dots, V_n$ is the standard open cover of $\mathbb{P}^n \subset \mathbb{P}^{2n+1}$. Here, the identification is

$$U_i \to V_i \times \mathbb{C}^{n+1}, \ [x_0, \dots, x_n, y_0, \dots, y_n] \mapsto ([x_0, \dots, x_n], (y_0/x_i, \dots, y_n/x_i)).$$

One checks that using this identification, we have

$$\begin{aligned} J(X, Y) \cap U_i &= \{[(tx, sy)] : [x] \in X, [y] \in Y, t \neq 0, x_i \neq 0\} \\ &= (X \cap V_i) \times \hat{Y} \subset V_i \times \mathbb{C}^{n+1} = U_i. \end{aligned}$$

As $X \cap V_i \subset V_i$ and $\hat{Y} \subset \mathbb{C}^{n+1}$ are closed, so is $J(X, Y) \cap U_i \subset U_i$. The case $n + 1 \leq i \leq 2n + 1$ is similar.

As for the dimension of $J(X, Y)$ we can see from the definition that its affine cone $\widehat{J(X, Y)} \subset \mathbb{C}^{2n+2} = \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ is given by $\hat{X} \times \hat{Y}$. By part b) and Theorem 2, we have

$$\begin{aligned} \dim(J(X, Y)) &= \dim(\widehat{J(X, Y)}) - 1 = \dim(\hat{X} \times \hat{Y}) - 1 \\ &= \dim(\hat{X}) + \dim(\hat{Y}) - 1 = \dim(X) + \dim(Y) + 1. \end{aligned}$$

e) We identify $\mathbb{P}^n \subset \mathbb{P}^{2n+1}$ as the space $D = \{[z_0, \dots, z_n, z_0, \dots, z_n]\}$. This is exactly the zero locus of the $n + 1$ linear forms $x_i - y_i$. It is clear that with this identification $X \cap Y = J(X, Y) \cap D$. On the other hand, using the assumptions we have $\dim(J(X, Y)) = \dim(X) + \dim(Y) + 1 \geq n + 1 = \operatorname{codim}(D)$, so by part c) this intersection is nonempty.

f) Take $Z = \mathbb{P}^1 \times \mathbb{P}^1$ and $X = \{[0, 1]\} \times \mathbb{P}^1$, $Y = \{[1, 0]\} \times \mathbb{P}^1$.

4. Prove that for any five lines $L_1, \dots, L_5 \subset \mathbb{P}^2$ in the projective plane there is a conic C tangent to all of them. Here we mean that for all i there is a point $p \in C \cap L_i$ such that $T_p L_i \subset T_p C$. *Hint:* Identify the space of plane conics with $\mathbb{P}^5$ and show that the set of conics tangent to a fixed line is a quadric hypersurface in $\mathbb{P}^5$.

Solution A conic in the plane is the zero-set of a homogeneous polynomial of degree 2, which has the form

$$f(x, y, z) = ax^2 + bxy + cxz + dy^2 + eyz + fz^2,$$

where $(a, b, c, d, e, f) \neq 0$. Using that $V(f) = V(\lambda f)$ for $\lambda \neq 0$, we identify the space of conics in the plane as $\mathbb{P}^5$ with homogeneous coordinates $[a, b, \dots, f]$. Now take a fixed line L and first assume we have $L = V(z) \cong \mathbb{P}^1$. Then a conic $V(f)$ is tangent to L iff the polynomial

$$f(x, y, 0) = ax^2 + bxy + dy^2$$

on $L \cong \mathbb{P}^1$ has a double zero. This is the case iff the discriminant $b^2 - 4ad$ vanishes. We see that this is a homogeneous quadratic equation in the coordinates on $\mathbb{P}^5$.

Now any line L' in $\mathbb{P}^2$ is related to L by a linear change of coordinates, which induces a linear change of coordinates on the space $\mathbb{P}^5$. Thus the locus of conics tangent to L' is still given as the zero locus of a homogeneous quadratic equation on $\mathbb{P}^5$. Let $\varphi_1, \dots, \varphi_5 \in \mathbb{C}[a, b, c, d, e, f]$ be such quadratic polynomials for the given lines $L_1, \dots, L_5$ and consider the Veronese embedding

$$\nu_2 : \mathbb{P}^5 \to \mathbb{P}^{14}.$$

Then the quadratic polynomials φ_i are induced from linear polynomials $u_1, \dots, u_5$ on $\mathbb{P}^{14}$ and $V(\varphi_i) = \nu_2^{-1}(V(u_i))$. To show that there exists a conic tangent to the L_i, it suffices to show that $\nu_2(\mathbb{P}^5) \cap V(u_1) \cap \dots \cap V(u_5) \neq \emptyset$. But as ν_2 is an embedding, $\dim(\nu_2(\mathbb{P}^5)) = \dim(\mathbb{P}^5) = 5$. On the other hand $V(u_1) \cap \dots \cap V(u_5)$ is a linear subspace of codimension less than or equal to 5. Hence by Exercise 3 c) its intersection with $\nu_2(\mathbb{P}^5)$ is indeed nonempty.

Due April 29.