1. Prove that the Zariski tangent space at the point $[S] \in \text{Gr}(r, V)$ is canonically isomorphic to $S^* \otimes V/S$ (or equivalently to $\text{Hom}(S, V/S)$).

**Solution** See e.g. Harris, *Algebraic Geometry*, Example 16.1

2. Let $M_{n,n}$ be the space of $n \times n$ complex matrices. Let $\det : M_{n,n} \to \mathbb{C}$ be the determinant and let $Z = V(\det)$ be its zero set. Find all the points of $Z$ such that the Zariski tangent space to $Z$ has dimension exactly $n^2 - 1$.

**Solution** Let $A \in M_{n,n}$ be a matrix and let $X_{kl}$ be coordinates on $M_{n,n}$. We want to compute the derivative of $\det$ with respect to the variable $X_{kl}$ at the point $A$. For this let $E_{kl} \in M_{n,n}$ be the matrix with coefficient 1 at the $(k,l)$-th position and 0 otherwise. By expanding $\det(A + tE_{kl})$ in the $k$-th column, we have

$$\frac{\partial \det}{\partial X_{kl}} = d(\det)_A(E_{kl}) = \frac{d}{dt} \bigg|_{t=0} \det(A + tE_{kl}) = \frac{d}{dt} \bigg|_{t=0} \sum_{j=1}^{n} (-1)^{k+j}(a_{kj} + \delta_{jl})M_{kj} = (-1)^{k+l}M_{kl}$$

where $M_{ij}$ is the $(i,j)$-th minor of $A$. In general we know that for $A \in Z$ the tangent space $T_A Z$ is cut out from $T_A M_{n,n} = \mathbb{C}^{n^2}$ by $d(\det)$, so it is of dimension at least $n^2 - 1$ with equality iff $d(\det) \neq 0$. Hence for $A \in V(\det)$,

$$d(\det) = 0 \iff \text{all partial derivatives of } \det \text{ vanish at } A$$

$$\iff M_{kl} = 0 \text{ for all } k, l$$

$$\iff \text{rank}(A) \leq n - 2$$

Hence the locus where the Zariski tangent space of $V(\det)$ has dimension $n^2 - 1$ are the matrices of rank exactly $n - 1$.

3. In this exercise, we want to show the following result about intersections of subvarieties of $\mathbb{P}^n$.

**Lemma 1.** Let $X, Y \subset \mathbb{P}^n$ be subvarieties such that $\dim(X) + \dim(Y) \geq n$. Then $X \cap Y \neq \emptyset$. 

**Exercise Sheet 7 - Solutions**
Below, we will first prove this result in the case that $Y$ is a linear subspace of $\mathbb{P}^n$ and then see how to show the general case from this.

a) For $X \subset \mathbb{P}^n$ a subvariety let

$$\hat{X} = \{ p \in \mathbb{C}^{n+1} \setminus \{0\} : [p] \in X \} \cup \{0\} \subset \mathbb{C}^{n+1}$$

be the affine cone over $X$. Show that $\hat{X}$ is an affine variety and that it is irreducible if $X$ is irreducible. Hint: For irreducibility, given open sets $V_1, V_2$ in $\hat{X}$, construct open sets in $X$ by intersecting with an affine hyperplane in $\mathbb{C}^{n+1}$.

b) Prove that $\dim(\hat{X}) = \dim(X) + 1$. Hint: Look at the intersections of $\hat{X}$ with $\hat{U}_i = \{(x_0, \ldots, x_n) : x_i \neq 0\}$.

c) Let $L \subset \mathbb{P}^n$ be a linear subspace of codimension $c$ (i.e. dimension $n - c$). Show that $\dim(X \cap L) \geq \dim(X) - c$. In particular, if $\dim(X) \geq c$ we have $X \cap L \neq \emptyset$. You may use the following result, a geometric version of Krull’s principal ideal theorem.

**Theorem 1.** If $X$ is an affine variety and $f$ is a regular function on $X$, then $\dim(V(f)) \geq \dim(X) - 1$.

d) Let $X, Y \subset \mathbb{P}^n$ be subvarieties. Consider $\mathbb{P}^{2n+1}$ with coordinates $x_0, \ldots, x_n, y_0, \ldots, y_n$ and identify $X \subset \mathbb{P}^n = V(y_0, \ldots, y_n) \subset \mathbb{P}^{2n+1}$ and similarly $Y \subset \mathbb{P}^n = V(x_0, \ldots, x_n) \subset \mathbb{P}^{2n+1}$. Consider the join $J(X, Y)$ of $X$ and $Y$ defined by

$$J(X, Y) = \{ [t(x, 0) + s(0, y)] : [x] \in X, [y] \in Y, (s, t) \in \mathbb{C} \setminus \{0\} \} \subset \mathbb{P}^{2n+1}.$$  

Show that $J(X, Y)$ is a subvariety of dimension $\dim(X) + \dim(Y) + 1$. Here you can use the following result, which will be proved later.

**Theorem 2.** If $X, Y$ are irreducible quasi-projective varieties, we have $\dim(X \times Y) = \dim(X) + \dim(Y)$.

e) Identify $X \cap Y$ as an intersection of $J(X, Y)$ with $n + 1$ hyperplanes in $\mathbb{P}^{2n+1}$. Conclude that it is nonempty if $\dim(X) + \dim(Y) \geq n$.

f) Give an example of a projective variety $Z$ with closed subvarieties $X, Y$ satisfying $\dim(X) + \dim(Y) \geq \dim(Z)$ but $X \cap Y = \emptyset$.

**Solution**

a) If $F_1, \ldots, F_r \in \mathbb{C}[x_0, \ldots, x_n]$ are homogeneous polynomials such that $X = V(F_1, \ldots, F_r)$, then $\hat{X} = V(F_1, \ldots, F_r) \subset \mathbb{C}^{n+1}$ and so indeed $\hat{X}$ is an algebraic variety.

In the case that $X$ is irreducible and let $V_1, V_2 \subset \hat{X}$ be nonempty open subsets. We want to show that they intersect.

Assume otherwise, then as for every $p \in \mathbb{C}^{n+1}$ the line $\mathbb{C} p \subset \mathbb{C}^{n+1}$ is irreducible, we must have $(\mathbb{C} p) \cap V_1 = \emptyset$ or $(\mathbb{C} p) \cap V_2 = \emptyset$. From this we see that we can choose $p_1 \in V_1, p_2 \in V_2$ not on a line through the origin. Then there exists an affine linear hyperplane $H \subset \mathbb{C}^{n+1}$ containing $p_1, p_2$
and not containing 0. The quotient map $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ restricted to $H$ defines an isomorphism of $H$ with an open subset of $\mathbb{P}^n$. A special case of this are the usual isomorphisms

$$H_i = \{(x_0, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)\} \to U_i \subset \mathbb{P}^n.$$ 

Intersecting with $\hat{X}$, we see that $\hat{X} \cap H$ is isomorphic to an open subset of $X$. The open subsets $V_i$ thus induce open subsets $V_i \cap \hat{X} \cap H$ in $X$ and these are nonempty because $p_i \in V_i \cap \hat{X} \cap H$. But we assumed that the $V_i$ do not intersect, giving a contradiction to the irreducibility of $X$.

b) To show the equality concerning the dimensions, we can consider all irreducible components of $X$ separately and therefore restrict to the case where $X$ is irreducible. Here we know by the previous exercise part that then also $\hat{X}$ is irreducible. To compute dimensions of $X, \hat{X}$, we will use that $\dim(Y) = \text{trdeg}_\mathbb{C}K(Y)$ for $Y$ an irreducible variety. Choose $i$ such that $X \cap U_i \neq \emptyset$, then for $R = \Gamma(X \cap U_i)$ the ring of functions on the affine variety $X \cap U_i$, we have

$$\dim(X) = \dim(X \cap U_i) = \text{trdeg}_\mathbb{C}Q(R),$$

where $Q(R)$ is the fraction field of the ring $R$. On the other hand, for the open set $\hat{U}_i = \{(x_0, \ldots, x_n) \in \mathbb{C}^{n+1} : x_i \neq 0\}$ we have

$$\dim(\hat{X}) = \dim(\hat{X} \cap \hat{U}_i) = \text{trdeg}_\mathbb{C}Q(\Gamma(\hat{X} \cap \hat{U}_i))$$

for the same reason. But we can identify the variety $\hat{X} \cap \hat{U}_i$: it is isomorphic to $(X \cap U_i) \times \mathbb{C}^*$ via the map

$$\hat{X} \cap \hat{U}_i \to (X \cap U_i) \times \mathbb{C}^*, (x_0, \ldots, x_n) \mapsto ([x_0, \ldots, x_n], x_i).$$

The inverse of this map is given by

$$([y_0, \ldots, y_n], \lambda) \mapsto (\lambda y_0/y_1, \ldots, \lambda y_{i-1}/y_i, \lambda, \lambda y_{i+1}/y_i, \ldots, \lambda y_n/y_i).$$

But as we have seen on a previous sheet

$$\Gamma((X \cap U_i) \times \mathbb{C}^*) = \Gamma(X \cap U_i) \otimes \Gamma(\mathbb{C}^*) = R \otimes \mathbb{C}[t, t^{-1}] = R[t, t^{-1}].$$

By elementary commutative algebra, we have

$$Q(R[t, t^{-1}]) = Q(R)(t),$$

so indeed $\text{trdeg}_\mathbb{C}(Q(R[t, t^{-1}])) = \text{trdeg}_\mathbb{C}(Q(R)) + 1$ as desired (if $y_1, \ldots, y_d$ are a transcencence basis of $Q(R)$, the elements $y_1, \ldots, y_d, t$ are a transcencence basis of $Q(R)(t)$).

c) It is clear that if we show the case of codimension $c = 1$, the general case follows by induction on $c$, as every linear subspace of codimension $c$ is an intersection of $c$ linear subspaces of codimension 1. So let $L = V(F)$ be a hyperplane, where $F$ is a linear form in $x_0, \ldots, x_n$. It is clear that the affine cone $\hat{X} \cap L$ of $X \cap L$ is exactly $\hat{X} \cap \hat{L}$, where
\( \hat{L} = V(F) \subset \mathbb{C}^{n+1} \) is the affine cone over \( L \), which is still the vanishing set of \( F \). But then by Krull’s principal ideal theorem, we have \( \dim(X \cap V(F)) \geq \dim(\hat{X}) - 1 \), so

\[
\dim(X \cap L) = \dim(X \cap V(F)) - 1 \geq \dim(\hat{X}) - 2 = \dim(X) - 1.
\]

Now let \( L \) again be any linear subspace of codimension \( c \). To show that \( X \cap L \) is nonempty if \( \dim(X) \geq c \) simply note that by the above argument, we showed \( \hat{X} \cap \hat{L} \) has dimension at least 1. Hence it cannot just be the set \( \{0\} \subset \mathbb{C}^{n+1} \) (which has dimension 0), but must contain a nonzero vector \( p \). Then \([p] \in X \cap L\).

d) We check that \( J(X,Y) \) is a closed subset of \( \mathbb{P}^{2n+1} \) on the standard open cover \( (U_i)_{i=0, \ldots, 2n+1} \) of \( \mathbb{P}^{2n+1} \). Assume \( 0 \leq i \leq n \), then we note that \( U_i = V_i \times \mathbb{C}^{n+1} \), where \( V_0, \ldots, V_n \) is the standard open cover of \( \mathbb{P}^n \subset \mathbb{P}^{2n+1} \). Here, the identification is

\[
U_i \to V_i \times \mathbb{C}^{n+1}, \quad [x_0, \ldots, x_n, y_0, \ldots, y_n] \mapsto ([x_0, \ldots, x_n], (y_0/x_i, \ldots, y_n/x_i)).
\]

One checks that using this identification, we have

\[
J(X,Y) \cap U_i = \{((tx, sy)) : [x] \in X, [y] \in Y, t \neq 0, x_i \neq 0\}
\]

\[
= (X \cap V_i) \times \hat{Y} \subset V_i \times \mathbb{C}^{n+1} = U_i.
\]

As \( X \cap V_i \subset V_i \) and \( \hat{Y} \subset \mathbb{C}^{n+1} \) are closed, so is \( J(X,Y) \cap U_i \subset U_i \). The case \( n+1 \leq i \leq 2n+1 \) is similar.

As for the dimension of \( J(X,Y) \) we can see from the definition that its affine cone \( \overline{J(X,Y)} \subset \mathbb{C}^{2n+2} = \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \) is given by \( \hat{X} \times \hat{Y} \). By part b) and Theorem 2, we have

\[
\dim(J(X,Y)) = \dim(\overline{J(X,Y)}) - 1 = \dim(\hat{X} \times \hat{Y}) - 1
\]

\[
= \dim(\hat{X}) + \dim(\hat{Y}) - 1 = \dim(X) + \dim(Y) + 1.
\]

e) We identify \( \mathbb{P}^n \subset \mathbb{P}^{2n+1} \) as the space \( D = \{[z_0, \ldots, z_n, z_0, \ldots, z_n]\} \). This is exactly the zero locus of the \( n+1 \) linear forms \( x_i - y_i \). It is clear that with this identification \( X \cap Y = J(X,Y) \cap D \). On the other hand, using the assumptions we have \( \dim(J(X,Y)) = \dim(X) + \dim(Y) + 1 \geq n + 1 = \text{codim}(D) \), so by part c) this intersection is nonempty.

f) Take \( Z = \mathbb{P}^1 \times \mathbb{P}^1 \) and \( X = \{[0,1]\} \times \mathbb{P}^1, Y = \{[1,0]\} \times \mathbb{P}^1 \).

4. Prove that for any five lines \( L_1, \ldots, L_5 \subset \mathbb{P}^2 \) in the projective plane there is a conic \( C \) tangent to all of them. Here we mean that for all \( i \) there is a point \( p \in C \cap L_i \) such that \( T_p L_i \subset T_p C \). Hint: Identify the space of plane conics with \( \mathbb{P}^5 \) and show that the set of conics tangent to a fixed line is a quadric hypersurface in \( \mathbb{P}^5 \).

Solution A conic in the plane is the zero-set of a homogeneous polynomial of degree 2, which has the form

\[
f(x,y,z) = ax^2 + bxy + cxy + dy^2 + eyz + f z^2,
\]
where \((a, b, c, d, e, f) \neq 0\). Using that \(V(f) = V(\lambda f)\) for \(\lambda \neq 0\), we identify the space of conics in the plane as \(\mathbb{P}^5\) with homogeneous coordinates \([a, b, \ldots, f]\).

Now take a fixed line \(L\) and first assume we have \(L = V(z) \cong \mathbb{P}^1\). Then a conic \(V(f)\) is tangent to \(L\) iff the polynomial

\[
f(x, y, 0) = ax^2 + bxy + dy^2
\]

on \(L \cong \mathbb{P}^1\) has a double zero. This is the case iff the discriminant \(b^2 - 4ad\) vanishes. We see that this is a homogeneous quadratic equation in the coordinates on \(\mathbb{P}^5\).

Now any line \(L'\) in \(\mathbb{P}^2\) is related to \(L\) by a linear change of coordinates, which induces a linear change of coordinates on the space \(\mathbb{P}^5\). Thus the locus of conics tangent to \(L'\) is still given as the zero locus of a homogeneous quadratic equation on \(\mathbb{P}^5\). Let \(\varphi_1, \ldots, \varphi_5 \in \mathbb{C}[a, b, c, d, e, f]\) be such quadratic polynomials for the given lines \(L_1, \ldots, L_5\) and consider the Veronese embedding

\[
\nu_2 : \mathbb{P}^5 \to \mathbb{P}^{14}.
\]

Then the quadratic polynomials \(\varphi_i\) are induced from linear polynomials \(u_1, \ldots, u_5\) on \(\mathbb{P}^{14}\) and \(V(\varphi_i) = \nu_2^{-1}(V(u_i))\). To show that there exists a conic tangent to the \(L_i\), it suffices to show that \(\nu_2(\mathbb{P}^5) \cap V(u_1) \cap \ldots \cap V(u_5) \neq \emptyset\). But as \(\nu_2\) is an embedding, \(\dim(\nu_2(\mathbb{P}^5)) = \dim(\mathbb{P}^5) = 5\). On the other hand \(V(u_1) \cap \ldots \cap V(u_5)\) is a linear subspace of codimension less than or equal to 5. Hence by Exercise 3 c) its intersection with \(\nu_2(\mathbb{P}^5)\) is indeed nonempty.

Due April 29.