Exercise Sheet 8 - Solutions

1. Consider the category of

(*) affine algebraic varieties with algebraic maps,

(**) finitely generated reduced $\mathbb{C}$-algebras with $\mathbb{C}$-algebra homomorphism.

Prove the equivalence of the categories (*) and (**) via the contravariant functor

$$X \to \Gamma(X)$$

Solution See Section 3 and in particular Corollary 3.8 in Hartshorne, Algebraic Geometry Chapter I.

2. Let X be an affine variety with coordinate ring A.

a) Show that X is irreducible iff A is a domain.

b) Recall that by the Nullstellensatz, the operations $Y \mapsto I(Y)$ and $I \mapsto V(I)$ define inverse bijections

$$\{\text{closed subsets } Y \subset X\} \leftrightarrow \{\text{radical ideals } I \subset A\}.$$

Show that this bijection restricts to bijections

$$\{\text{irreducible closed subsets } Y \subset X\} \leftrightarrow \{\text{prime ideals } I \subset A\},$$
$$\{\text{points } p \in X\} \leftrightarrow \{\text{maximal ideals } m \subset A\}.$$

Solution

a) Assume first that X is irreducible and let $f, g \in A$ with $fg = 0$. Then $X = V(fg) = V(f) \cup V(g)$ is the union of the closed sets $V(f), V(g)$. As X is irreducible, this means that $V(f) = X$ or $V(g) = X$. Assume the first, then as $V(f) = X = V(0)$, by the Nullstellensatz we have $f^n = 0$ for some n, hence $f = 0$.
Conversely, assume that A is a domain and let $X = V(I) \cup V(J)$ be the union of two closed sets for ideals $I, J \subset A$. Then for $f \in I, g \in J$ we have $V(I) \subset V(f)$ and $V(J) \subset V(g)$, thus $X = V(f) \cup V(g) = V(fg)$. Again by the Nullstellensatz we have $(fg)^n = 0$, hence $fg = 0$. But as A is a domain, we know either $f = 0$ or $g = 0$. As f, g were arbitrary in I, J, we must have $I = 0$ or $J = 0$, so indeed $V(I) = X$ or $V(J) = X$.

b) Recall that the coordinate ring of $V(I) \subset X$ is given by A/I. Thus by the previous exercise, $V(I)$ is irreducible iff A/I is a domain. But this is the case iff I is prime. On the other hand, if $V(I)$ is a point, then clearly $\Gamma(V(I)) = \mathbb{C}$ is a field, so I is a maximal ideal. On the other hand, for a maximal ideal m look at $V = V(m)$. Clearly $V \neq \emptyset$ as otherwise $V(m) = V(A)$, so there exists n with $1^n = 1 \in m$, a contradiction. But for $p \in V$ we clearly have $m = I(V(m)) \subset I(\{p\})$, so as m is maximal we have $m = I(\{p\})$ and so the vanishing set $V(m) = \{p\}$ is indeed a point.

3. Let X be a quasi-projective variety and $p \in X$. Define

$$\mathcal{O}_{X,p} = \{(f, U) : p \in U \subset X \text{ nonempty open}, f : U \to \mathbb{C} \text{ algebraic}\}/\sim,$$

where $(f, U) \sim (g, V)$ if there exists a nonempty open neighborhood $W \subset U \cap V$ of p with $f|_W = g|_W$. Show that $\mathcal{O}_{X,p}$ has a natural $\mathbb{C}$-algebra structure. It is called the *ring of germs of algebraic functions* on X around p. If $V \subset X$ is an open neighborhood of p, show that there is a natural isomorphism $\mathcal{O}_{X,p} \cong \mathcal{O}_{V,p}$. If V is affine with coordinate ring A and if $p \in V$ corresponds to the maximal ideal $m \subset A$, show that $\mathcal{O}_{V,p}$ is isomorphic to the localization A_m.

Solution The algebra structure on $\mathcal{O}_{X,p}$ is given by

$$\begin{aligned}
[(f, U)] + [(f', U')] &= [(f|_{U\cap U'} + f'|_{U\cap U'}, U \cap U')], \\
[(f, U)] \cdot [(f', U')] &= [(f|_{U\cap U'} \cdot f'|_{U\cap U'}, U \cap U')], \\
\lambda[(f, U)] &= [(\lambda f, U)] \quad (\lambda \in \mathbb{C}).
\end{aligned}$$

One checks that this is well-defined (i.e. does not depend on the choice of representatives $(f, U), (f', U')$), because addition and multiplication commute with restriction of functions.

If $V \subset X$ is an open neighborhood of p, the map

$$\mathcal{O}_{X,p} \to \mathcal{O}_{V,p}, [(f, U)] \mapsto [(f|_{U\cap V}, U \cap V)]$$

defines an isomorphism with inverse $[(f', U')] \mapsto [(f', U')]$ (think about what this means).

For an affine V with coordinate ring A, such that $m = I(\{p\}) \subset A$ is the maximal ideal corresponding to p, we define a map $A_m \to \mathcal{O}_{V,p}$ by noting that an element $f/g \in A_m$ ($f \in A, g \in A\setminus m$) defines a function on the neighborhood $D(g) = \{q \in V : g \neq 0\} \subset X$ of p. Indeed, by assumption $g \notin m$, so $g(p) \neq 0$, hence $p \in D(g)$. Thus we define

$$A_m \to \mathcal{O}_{V,p}, f/g \mapsto [(f/g, D(g))].$$

This is well-defined: if we choose a different representative $f'/g' = f/g$ in A_m, then by definition there exists $u \in A \setminus m$ with $u(f'g - fg') = 0 \in A$. Then restricted to the open neighborhood $D(u) \cap D(g) \cap D(g')$ of p we have indeed $f'/g' = f/g$.

To see injectivity note that $f/g = 0$ on $D(g)$ iff $fg = 0$ as functions on V, thus $fg = 0 \in A$. But as $g \notin m$, this means $f/g = 0 \in A_m$.

For surjectivity, assume we have a function h on a neighborhood U of p, then as the sets $D(g)$, $g \in A$, form a basis of the Zariski topology, we can find g with $p \in D(g) \subset U$. But then $[(h, U)] = [(h|_{D(g)}, D(g))]$. As we have seen, the functions on $D(g)$ are exactly the localization A_g, so h can be written as $h = f/g^m$ and $g \notin m$. This comes exactly from $f/g^m \in A_m$ by the above map, finishing the proof.

4. Let X be an affine algebraic variety and let A be the ring of algebraic functions on X. Let $p \in X$ be a point and let $\mathfrak{m} \subset A$ be the associated maximal ideal. Let $A_\mathfrak{m}$ be the localization of A at $\mathfrak{m}$. Let $\mathfrak{m}A_\mathfrak{m}$ be the maximal ideal of $A_\mathfrak{m}$. Prove that the natural map

$$\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}A_\mathfrak{m}/\mathfrak{m}^2 A_\mathfrak{m}$$

is an isomorphism of $A/\mathfrak{m}$ vector spaces.

Solution Let $\mathfrak{p} \subset A$ be a prime ideal of a ring A. Let $\varphi : A \to A_\mathfrak{p}$ be the localization. For any ideal $I \subset A$, we write $IA_\mathfrak{p}$ for $\varphi(I) \cdot A_\mathfrak{p}$, which is an ideal of $A_\mathfrak{p}$. Let M be any A-module, then the localization of M along $\mathfrak{p}$ is denoted $M_\mathfrak{p}$. It is isomorphic to $M \otimes_A A_\mathfrak{p}$.

Let

$$0 \to I \to A \to A/I \to 0$$

be an exact sequence for an ideal I. As localizing at a prime ideal $\mathfrak{p}$ is exact, we obtain the exact sequence,

$$0 \to I_\mathfrak{p} \to A_\mathfrak{p} \to (A/I)_\mathfrak{p} \to 0.$$

Thus we get $(A/I)_\mathfrak{p} = A_\mathfrak{p}/I_\mathfrak{p}$. Furthermore, it is easy to show that the kernel of $A_\mathfrak{p} \to A_\mathfrak{p}/I_\mathfrak{p}$ is $IA_\mathfrak{p}$, hence also $IA_\mathfrak{p} \cong I_\mathfrak{p} = I \otimes A_\mathfrak{p}$.

For our problem, consider the exact sequence

$$0 \to \mathfrak{m}^2 \to \mathfrak{m} \to \mathfrak{m}/\mathfrak{m}^2 \to 0$$

Localizing/Tensoring with $A_\mathfrak{m}$ and using $IA_\mathfrak{p} \cong I \otimes A_\mathfrak{p}$ from above twice, we obtain

$$0 \to \mathfrak{m}^2 A_\mathfrak{m} \to \mathfrak{m}A_\mathfrak{m} \to (\mathfrak{m}/\mathfrak{m}^2) \otimes A_\mathfrak{m} \to 0$$

which implies $(\mathfrak{m}/\mathfrak{m}^2) \otimes A_\mathfrak{m} \cong \mathfrak{m}A_\mathfrak{m}/\mathfrak{m}^2 A_\mathfrak{m}$. To conclude, note that

$$(\mathfrak{m}/\mathfrak{m}^2) \otimes A_\mathfrak{m} = (\mathfrak{m}/\mathfrak{m}^2) \otimes A_\mathfrak{m}/\mathfrak{m}A_\mathfrak{m} = \mathfrak{m}/\mathfrak{m}^2 \otimes A/\mathfrak{m} = \mathfrak{m}/\mathfrak{m}^2 \otimes A \cong \mathfrak{m}/\mathfrak{m}^2$$

Here the first and third equality use that if M is a A-module, and $I \subset A$ an ideal such that $I \cdot M = 0$, then $M \otimes_A N = M \otimes (N/IN)$ for all A-modules N. Also we use $A_\mathfrak{m}/\mathfrak{m}A_\mathfrak{m} = (A/\mathfrak{m})_\mathfrak{m} = A/\mathfrak{m}$, where we note that $\mathfrak{m} \subset A/\mathfrak{m}$ is the zero-ideal in the field $A/\mathfrak{m}$, so localizing here does not change $A/\mathfrak{m}$.

5. Let X and Y be irreducible quasi-projective varieties. Recall that X and Y are birational if there are nonempty open sets $U \subset X$, $W \subset Y$, such that U is isomorphic to W. Prove that X and Y are birational if and only if $K(X)$ is isomorphic to $K(Y)$.

Solution As the field of rational functions of a variety depends only on an open set and two (irreducible) varieties are birational if and only if they are birational on an open set, we can assume that X and Y are affine with coordinate rings $A = k[x_1, \ldots, x_n]/I$ and $B = k[y_1, \ldots, y_m]/J$ respectively. Clearly birational implies isomorphic rings of rational functions so let us assume that we have an isomorphism $\varphi : Q(A) \to Q(B)$ of the quotient fields with inverse $\psi : Q(B) \to Q(A)$.

Let $\varphi(x_i) = \frac{b_i}{g_i}$, $i = 1, \ldots, n$ and $\psi(y_i) = \frac{a_i}{f_i}$, $i = 1, \ldots, m$. This defines regular maps

$$\begin{aligned} F : D(f_1) \cap \cdots \cap D(f_m) \subset X \to Y, &\quad P \mapsto (a_1(P)/f_1(P), \ldots, a_m(P)/f_m(P)) \\ G : D(g_1) \cap \cdots \cap D(g_n) \subset Y \to X, &\quad P \mapsto (b_1(P)/g_1(P), \ldots, a_n(P)/g_n(P)) \end{aligned}$$

The ring homomorphism $\psi|_B : B \to A[1/f_1, \ldots, 1/f_n]$ is injective, hence the map F has a dense image (if not, then there would be an open subset $V \subset Y \setminus \mathrm{im}(F)$ and an element $g \in B$ with $D(g) \subset V$. This would imply $\psi(g) = g \circ F = 0$.)

As φ and ψ are inverse to each other, we have $G \circ F = id$ and $F \circ G = id$ whenever they are defined. Let $U = D(f_1) \cap \cdots \cap D(f_m)$ and $V = D(g_1) \cap \cdots \cap D(g_n)$. Then $G \circ F(x)$ is defined when $x \in U$ and $F(x) \in V$, or in other words $G \circ F$ is defined on $U \cap F^{-1}(V)$, and similarly $F \circ G$ is defined on $V \cap G^{-1}(U)$. Let

$$\widetilde{U} = U \cap F^{-1}(V), \qquad \widetilde{V} = V \cap G^{-1}(U)$$

and consider the restricted maps $F = F|_{\widetilde{U}} : \widetilde{U} \to Y$ and $G = G|_{\widetilde{V}}$. By the argument above we have seen, that both $\widetilde{U}$ and $\widetilde{V}$ are non-empty.

Claim: $F : \widetilde{U} \to \widetilde{V}$.

Proof of Claim: Let $x \in \widetilde{U}$. Then $x \in U$ and $F(x) \in V$. If $F(x) \in G^{-1}(U)$, then $F(x) \in \widetilde{V}$ and we are done. But $F(x) \in G^{-1}(U)$ is equivalent to $G(F(x)) \in U$, which is true since $G(F(x)) = x$. Hence $F : \widetilde{U} \to \widetilde{V}$.

Analogously, we have $G : \widetilde{V} \to \widetilde{U}$. But now $G \circ F$ is defined on $\widetilde{U}$, and $F \circ G$ on $\widetilde{V}$, with $G \circ F = id_{\widetilde{U}}$, $F \circ G = id_{\widetilde{V}}$. Hence $\widetilde{U} \cong \widetilde{V}$ and we are done.

Due May 06.