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Algebraic Topology II

FS 2016

Problem set 4

1. Show that every covering space of an orientable manifold is an orientable manifold.

2. Show that for a connected non-orientable manifold M there is a unique orientable double cover of M.

3. Show that for any connected closed orientable n-manifold M there is a degree 1 map $f : M \to S^n$.

4. Let $f : M \to N$ be a map between connected closed orientable manifolds and suppose there is a ball $B \subset N$ such that $f^{-1}(B)$ is a disjoint union of open balls $B_1, \ldots, B_k \subset M$ which each get mapped homeomorphically onto B. Show that the degree of f is $\sum \varepsilon_i$, where ε_i is ± 1 according to whether $f|_{B_i} : B_i \to B$ preserves or reverses local orientations induced from given fundamental classes $[M]$ and $[N]$.

5. Let M, N be closed connected orientable manifolds and let $f : M \to N$ a p-sheeted covering map. Show that f has degree $\pm p$.

6. Consider a pair of spaces $(X, Y) = (Q \cup R, S \cup T)$ such that $S \subset Q$, $T \subset R$ and such that the interiors of Q, R cover X and the interiors of S, T cover Y. Show that there is a relative Mayer-Vietoris LES

   $$\cdots \to H_n(Q \cap R, S \cap T) \to H_n(Q, S) \oplus H_n(R, T) \to H_n(X, Y) \to H_{n-1}(Q \cap R, S \cap T) \to \cdots$$

   Hint: Consider the commutative diagram

   $$\begin{array}{ccccccccc}
    & & 0 & & 0 & & 0 & & \\
    & & \downarrow & & \downarrow & & \downarrow & & \\
   0 & \longrightarrow & S_n(S \cap T) & \longrightarrow & S_n(S) \oplus S_n(T) & \longrightarrow & S_n(S+T) & \longrightarrow & 0 \\
    & & \downarrow & & \downarrow & & \downarrow & & \\
   0 & \longrightarrow & S_n(Q \cap R) & \longrightarrow & S_n(Q) \oplus S_n(R) & \longrightarrow & S_n(Q+R) & \longrightarrow & 0 \\
    & & \downarrow & & \downarrow & & \downarrow & & \\
   0 & \longrightarrow & S_n(Q \cap R, S \cap T) & \longrightarrow & S_n(Q, S) \oplus S_n(R, T) & \longrightarrow & S_n(Q+R, S+T) & \longrightarrow & 0 \\
    & & \downarrow & & \downarrow & & \downarrow & & \\
    & & 0 & & 0 & & 0 & &
   \end{array}$$

   in which the horizontal maps are of the form $x \mapsto (x, -x)$ resp. $(x, y) \mapsto x + y$; $S_n(Q+R)$ is the subgroup of $S_n(X)$ consisting of sums of chains in Q and R (and similarly for $S_n(S+T)$), and $S_n(Q+R, S+T)$ denotes the quotient of $S_n(Q+R)$ by $S_n(S+T)$. Show first that the third row is a chain complex. Then show it is exact by considering the diagram as a short exact sequence of chain complexes. Finally deduce the existence of the LES.