Problem set 5

1. Show that for closed connected n-manifolds M_1, M_2 there are isomorphisms $H_i(M_2) \oplus H_i(M_2) \cong H_i(M_1 \# M_2)$ for $0 < i < n$ with one exception: If both M_1 and M_2 are non-orientable, then $H_{n-1}(M_1 \# M_2)$ is obtained from $H_{n-1}(M_1) \oplus H_{n-1}(M_2)$ by replacing one the $\mathbb{Z}_2$ summands by a $\mathbb{Z}$-summand. *Hint:* Euler characteristics may help in the exceptional case. (For the definition of the connected sum $M_1 \# M_2$ see algebraic topology I problem set 5 ex. 6.)

2. Show that if a closed orientable manifold of dimension $2n$ has $H_{n-1}(M)$ torsion-free then $H_n(M)$ is also torsion-free.

3. Compute the cup product structure of $H^*((S^2 \times S^8) \# (S^4 \times S^6))$, and in particular show that the only non-trivial cup products are those forced by Poincaré duality.

4. Show that if M is a compact connected non-orientable 3-manifold, $H_1(M)$ is infinite.

5. Prove that every map $f : \mathbb{C}P^n \to \mathbb{C}P^n$ has $\deg f = k^n$ for some $k \in \mathbb{Z}$.

6. Let $\alpha \in H^n(S^n)$ be a generator, and define $u = \alpha \times 1, v = 1 \times \alpha \in H^n(S^n \times S^n)$. Let now $f : S^n \times S^n \to S^n \times S^n$ be a map with $\deg f = \pm 1$. Writing $f^*(u) = au + bv$, $f^*v = cu + dv$ and assuming that n is even, prove that
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}.$$

7. Let M be a closed connected orientable n-manifold and suppose that there exists a map $f : S^n \to M$ with $\deg f \neq 0$. Prove that $H_*(M; \mathbb{Q}) \cong H_*(S^n; \mathbb{Q})$. If $\deg f = \pm 1$, prove that $H_*(M; \mathbb{Z}) \cong H_*(S^n; \mathbb{Z})$.

8. Prove that if a closed connected orientable manifold M can be written as the union $M = U \cup V$ of two acyclic subsets, then $H_*(M) \cong H_*(S^n)$. *Hint:* Use problem 3/3.