ETH Zürich FS 2016
D-MATH
Prof. Pierre Nolin

Coordinator
Wei Qian

Brownian Motion and Stochastic Calculus
Exercise Sheet 1

Please hand in until Friday, March 11th, in your exercise group and otherwise before 13:00, in HG E 66.1

Exercise 1-1

Let X be a real valued random variable with standard normal distribution as law and Y a random variable independent of X with law defined by

$$P[Y = 1] = p \quad \text{and} \quad P[Y = -1] = 1 - p, \qquad (0 \le p \le 1).$$

We define $Z := XY$.

a) What is the law of Z? Is the vector (X, Z) a Gaussian vector?

b) Calculate $\mathrm{Cov}(X, Z)$. For which $p \in [0, 1]$ are the random variables X and Z uncorrelated, i.e. $\mathrm{Cov}(X, Z) = 0$?

c) Show that for each $p \in [0, 1]$, the random variables X and Z are **not** independent.

Exercise 1-2

Let W be a Brownian motion on $[0, 1]$ and define the *Brownian bridge* $X = (X_t)_{0 \le t \le 1}$ by $X_t = W_t - tW_1$.

a) Show that X is a Gaussian process and calculate its mean and covariance functions. Sketch a typical path of X.

b) Show that X does **not** have independent increments.

Exercise 1-3

Let $(\Omega, \mathcal{F}, P)$ be a probability space and assume that $X = (X_t)_{t \ge 0}$, $Y = (Y_t)_{t \ge 0}$ are two stochastic processes on $(\Omega, \mathcal{F}, P)$. Two processes Z and Z' on $(\Omega, \mathcal{F}, P)$ are said to be *modifications* of each other if $P(Z_t = Z'_t) = 1 \ \forall\, t \ge 0$, while Z and Z' are *indistinguishable* if $P(Z_t = Z'_t \ \forall\, t \ge 0) = 1$.

a) Assume that X and Y are both right-continuous or both left-continuous. Show that the processes are modifications of each other if and only if they are indistinguishable.

Remark: A stochastic process is said to *have the path property* $\mathcal{P}$ ($\mathcal{P}$ can be continuity, right-continuity, differentiability, ...) if the property $\mathcal{P}$ holds for P-almost every path.

b) Give an example showing that one of the implications of part **a)** does not hold for general X, Y.
