ETH Zürich FS 2016
D-MATH
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Brownian Motion and Stochastic Calculus
Exercise Sheet 10

Please hand in until Friday, May 27th, in your exercise group and otherwise before 13:00, in HG E 66.1

Exercise 10-1

Let $(B_t)_{t\geq 0}$ be a Brownian motion and let $(X_t)_{t\geq 0}$ be defined by $X_t = \int_0^t \operatorname{sign}(B_s)\, dB_s$, where $\operatorname{sign}(x) = 1$ for $x \geq 0$ and $\operatorname{sign}(x) = -1$ for $x < 0$.

a) Show that $(X_t)_{t\geq 0}$ is a Brownian motion and that $E[X_t\, B_s] = 0$ for all $s, t \geq 0$ (which means that X and B are uncorrelated).

b) Show that $E[X_t B_t^2] = 2^{\frac{5}{2}} t^{\frac{3}{2}} \frac{1}{3\sqrt{\pi}}$ and conclude that $(X_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$ are not independent (despite being uncorrelated and Gaussian processes).

Exercise 10-2

Let X be the canonical Brownian motion in $\mathbb{R}^d, d \geq 2$, starting from $x \neq 0$ (by (6.50) one knows that W_x-a.s., $X_t \neq 0$ for all $t \geq 0$). Set $R_t = |X_t|, t \geq 0$.

a) Show that W_x-a.s., for $t \geq 0$,

$$R_t = |x| + \sum_{i=1}^{d} \int_0^t \frac{X_s^i}{|X_s|}\, dX_s^i + \frac{(d-1)}{2} \int_0^t \frac{1}{|X_s|}\, ds.$$

b) Show that there is a one dimensional Brownian motion B so that

$$W_x\text{-a.s., for all } t \geq 0,\ R_t = |x| + B_t + \frac{(d-1)}{2} \int_0^t \frac{1}{R_s}\, ds$$

(R is the d-dimensional Bessel process).

Exercise 10-3

Let $(B_t)_{t\geq 0}$ be a Brownian motion and consider the stochastic differential equation (SDE)

$$dX_t = -\gamma X_t dt + \sigma dB_t, \quad X_0 = x. \tag{1}$$

A solution of the SDE in (1) is called *Ornstein-Uhlenbeck process*.

a) Show that the unique solution (starting from x) is given explicitly by

$$X_t^x = e^{-\gamma t} \left(x + \sigma \int_0^t e^{\gamma s} dB_s \right).$$

b) Show that for any $t > 0$, $X_t^x \sim \mathcal{N}\left(e^{-\gamma t}x, \frac{\sigma^2}{2\gamma}(1 - e^{-2\gamma t})\right)$.
Hint: Use Series 7 Exercise 2.
