ETH Zürich FS 2016
D-MATH
Prof. Pierre Nolin

Coordinator
Wei Qian

# Brownian Motion and Stochastic Calculus
## Exercise Sheet 11

*Please hand in until Friday, June 3rd, in your exercise group*
*and otherwise before 13:00, in HG E 66.1*

### Exercise 11-1

Let $(B_t)_{t\geq 0}$ be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$. Consider the SDE

$$dX_t = \Big(\sqrt{1+X_t^2} + \frac{1}{2}X_t\Big)\,dt + \sqrt{1+X_t^2}\,dB_t, \quad X_0 = x \in \mathbb{R}. \tag{1}$$

**a)** Show that for any $x \in \mathbb{R}$ the SDE defined in (1) has a unique strong solution.

**b)** Show that $(X_t)_{t\geq 0}$ defined by $X_t = \sinh\big(\operatorname{arsinh}(x) + t + B_t\big)$ is the unique solution of (1).
**Hint:** Consider the process $(Y_t)_{t\geq 0}$ defined by $Y_t := \operatorname{arsinh}(X_t)$.

### Exercise 11-2

Let $(B_t)_{t\geq 0}$ be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$ and $(X_t)_{t\geq 0}$ the unique solution of the SDE

$$dX_t = f(X_t)\,dt + g(X_t)\,dB_t, \quad X_0 = x, \tag{2}$$

where $f, g : \mathbb{R} \to \mathbb{R}$ are Lipschitz-continuous functions.

**a)** Find a non-constant function $\phi(x) \in C^2(\mathbb{R}, \mathbb{R})$ such that $Y_t := \phi(X_t)$ is a local martingale. Moreover, derive a SDE for $(Y_t)_{t\geq 0}$.
**Hint:** Use that the general solution of the ODE: $y' f(x) + \frac{1}{2} y'' g^2(x) = 0$ is of the form

$$y(x) = a + b\int_0^x \exp\Big(-2\int_0^u \frac{f(v)}{g^2(v)}\,dv\Big)\,du, \quad a, b \in \mathbb{R}.$$

**b)** Assume additionally that $f$ is negative on $(-\infty, 0)$ and positive on $[0, \infty)$. Show that in this case, $Y$ is a martingale.

**Hint:** Use Series 9 Exercise 1. Also, we can assume that if $(X_t)_{t\geq 0}$ is the strong solution of (2), then for all $T > 0$, we have $E\left[\sup_{0\leq t\leq T} |X_t|^2\right] < \infty$. This can be seen from the proof of Theorem 7.1 of the lecture notes.

### Exercise 11-3

Let $(B_t)_{t\geq 0}$ be a Brownian motion defined on a filtered probability space $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P\big)$. Moreover, let $\xi \in L^2(\Omega, \mathcal{F}_0, P)$. Consider the SDE given by

$$dX_t = r_t X_t\,dt + v_t X_t\,dB_t, \quad X_0 = \xi \tag{3}$$

where $r, v : [0, \infty) \to \mathbb{R}$ are bounded measurable deterministic functions.
Show that the SDE defined in (3) has a unique strong solution. Moreover give the explicit solution by writing *formally* the stochastic differential equation satisfied by $Y := \log(X)$.

Exercise sheets and further information are also available on:

**http://www.math.ethz.ch/education/bachelor/lectures/fs2016/math/bmsc**