ETH Zürich FS 2016
D-MATH
Prof. Pierre Nolin

Coordinator
Wei Qian

Brownian Motion and Stochastic Calculus
Optional Exercise Sheet 12

Solution will be posted on line Friday, June 3rd, after 13:00

Exercise 12-1

Let $(B_t)_{t\geq 0}$ be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$. Consider the SDE

$$X_t = \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\,dB_s, \quad X_0 = 0 \tag{1}$$

where $b(x) := 3x^{1/3}$ and $\sigma(x) := 3x^{2/3}$. Show that the SDE has uncountably many strong solutions of the form

$$X_t^{(\Theta)} = \begin{cases} 0, & 0 \leq t < \beta_\Theta, \\ B_t^3, & \beta_\Theta \leq t < \infty, \end{cases}$$

where $0 \leq \Theta \leq \infty$ is any fixed constant and $\beta_\Theta := \inf\{s \geq \Theta \,|\, B_s = 0\}$.

Exercise 12-2

Let $(W_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$ two independent Brownian motions defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$. Consider a market where we have two assets whose prices follow the SDEs:

$$dS_t = S_t(\mu dt + \sigma dW_t), \quad S_0 > 0,$$
$$dX_t = X_t(b\,dt + a\,dB_t), \quad X_0 > 0,$$

where $\mu, b \in \mathbb{R}$ and $\sigma, a > 0$.

a) Find the SDEs satisfied by $\widetilde{X} = \frac{X}{S}$ and by $\widetilde{S} = \frac{S}{X}$.
Hint: We can admit that $S_t, X_t > 0$ for all $t \geq 0$ P-a.s., so that both processes $(\widetilde{X}_t)_{t\geq 0}$ and $(\widetilde{S}_t)_{t\geq 0}$ are well defined. (This fact can be shown by explicitly solving the equation.)

b) For which values of σ, μ, a, b are the processes $(\widetilde{S}_t)_{t\geq 0}$ and $(\widetilde{X}_t)_{t\geq 0}$ martingales, submartingales, supermartingales with respect to P and $(F_t)_{t\geq 0}$?

Exercise 12-3

Let $b(\cdot) \geq 0$ be bounded measurable on $\mathbb{R}$ and let P_{x_0} be the unique solution of the martingale problem (L, x_0) on $\big(C(\mathbb{R}_+, \mathbb{R}), \mathcal{F}\big)$ as in (7.2) in the notes, but where $L := \frac{1}{2}\frac{\partial^2}{\partial x^2} + \big(1 + b(x)\big)\frac{\partial}{\partial x}$, i.e. for all $f \in C_c^2(\mathbb{R})$

$$Lf(x) := \frac{1}{2}\frac{\partial^2}{\partial x^2} f(x) + \big(1 + b(x)\big)\frac{\partial}{\partial x} f(x).$$

If W_{x_0} stands for the Wiener measure starting at x_0, show that P_{x_0} is absolutely continuous with respect to W_{x_0} on $\mathcal{F}_T$ for each $T > 0$, (where $(\mathcal{F}_t)_{t\geq 0}$ is the filtration generated by the canonical process X), but there is a set $A \in \mathcal{F}$ such that $W_{x_0}(A) = 1$ and $P_{x_0}(A^c) = 1$ (i.e. W_{x_0}

and P_{x_0} are mutually singular).
Hint: Use Girsanov transformation. Consider $A = \{\omega | \limsup_{t\to\infty} X_t(\omega)/t = 0\}$.

Exercise 12-4

Consider the boundary problem

$$\begin{cases} Lu(x) + c(x)u(x) & = -f(x), \text{ for } x \in U, \\ u(x) & = g(x), \text{ for } x \in \partial U, \end{cases}$$

where $U \neq \emptyset$ is a bounded open subset on $\mathbb{R}^d$, $f \in C_b(U)$, $g \in C_b(\partial U)$, $c \leq 0$ is a uniformly bounded function on $\mathbb{R}^d$ and L is defined as in (7.25). Show that if $u \in C^2(U) \cap C(\bar{U})$ is a solution of the above boundary problem and $(X_t)_{t\geq 0}$ satisfies (7.23), for some $x \in U$, then

$$u(x) = E\Big[g(X_{T_U}) \exp\Big(\int_0^{T_U} c(X_s)\,ds\Big)\Big] + E\Big[\int_0^{T_U} f(X_s) \exp\Big(\int_0^s c(X_r)\,dr\Big)\,ds\Big],$$

where $T_U := \inf\{s \geq 0; X_s \notin U\}$.
Hint: Argue similarly as on page 114-115 in the lecture notes (compare with (7.32)).
