ETH Zürich FS 2016
D-MATH
Prof. Pierre Nolin

Coordinator
Wei Qian

Brownian Motion and Stochastic Calculus
Exercise Sheet 2

Please hand in until Tuesday, March 18th, in your exercise group and otherwise before 13:00, in HG E 66.1

Exercise 2-1

Let $(B_t)_{t\geq 0}$ be a Brownian motion and consider the Ornstein-Uhlenbeck diffusion given by

$$X_t := e^{-t} B_{e^{2t}}\,, \quad t \in \mathbb{R}.$$

a) Show that $X_t \sim \mathcal{N}(0,1), \quad \forall t \in \mathbb{R}$.

b) Show that the process $(X_t)_{t\in\mathbb{R}}$ is *time reversible*, i.e. $(X_t)_{t\geq 0} \stackrel{Law}{=} (X_{-t})_{t\geq 0}$.

Exercise 2-2

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables with $X_n \sim \mathcal{N}(\mu_n, \sigma_n^2)$ for each $n \in \mathbb{N}$.

a) Show that if the sequence $(X_n)_{n\in\mathbb{N}}$ converges in distribution to a random variable X, then the limits $\mu := \lim_{n\to\infty} \mu_n$ and $\sigma^2 := \lim_{n\to\infty} \sigma_n^2$ exist and $X \sim \mathcal{N}(\mu, \sigma^2)$.

b) Show that if $(X_n)_{n\in\mathbb{N}}$ is a Gaussian process indexed by $\mathbb{N}$ and converges in probability to a random variable X as n goes to infinity, then it converges also in L^2 to X.

Exercise 2-3

Let $X = (X_t)_{t\geq 0}$ be the canonical process ("the canonical coordinates") defined on the canonical space $(\Omega, \mathcal{F}) = (C[0,\infty), \sigma(X_u,\, u \geq 0))$ for Brownian motion and let $(\mathcal{F}_t)_{t\geq 0}$ be the filtration generated by X. For each $x \in \mathbb{R}$, let W_x be the Wiener measure starting from x. Denote by $\widehat{\mathcal{F}} := \bigcap_{t\geq 0} \widehat{\mathcal{F}}_t$ the *asymptotic σ-field*, where, $\widehat{\mathcal{F}}_t := \sigma(X_u,\, u \geq t)$ for $t \geq 0$.

a) Use the Blumenthal 0-1 law and the invariance by time inversion property of Brownian motion to conclude that $W_0[A] = 0$ or $W_0[A] = 1$ for every $A \in \widehat{\mathcal{F}}$.

b) Conclude that for each $A \in \widehat{\mathcal{F}}$, either $W_x[A] = 0$ for each $x \in \mathbb{R}$ or $W_x[A] = 1$ for each $x \in \mathbb{R}$.

Hint: For any $A \in \widehat{\mathcal{F}}_1$, write $1_A = 1_{A'} \circ \theta_1$ for some $A' \in \mathcal{F}$ and use the Markov property.
