ETH Zürich FS 2016
D-MATH
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Brownian Motion and Stochastic Calculus
Exercise Sheet 3

Please hand in until Friday, April 1st, 13:00, in the assistants' boxes in HG E 66.1.

Exercise 3-1

Let $(B_t)_{t\geq 0}$ be a Brownian motion and denote by $\mathcal{G}_t := \sigma(B_u,\, u \leq t)$, $t \geq 0$.
Define $\widetilde{R}_0 f(x) = f(x)$ and

$$\widetilde{R}_t f(x) = \frac{1}{\sqrt{2\pi t}} \int_0^\infty f(y) \left[\exp\left(-\frac{1}{2t}(y-x)^2\right) + \exp\left(-\frac{1}{2t}(y+x)^2\right)\right] dy, \quad t > 0$$

Define the process $(X_t)_{t\geq 0}$ by $X_t := |B_t|$.
Show that

$$E\big[f(X_{t+h}) \,\big|\, \mathcal{G}_t\big] = \widetilde{R}_h f(X_t) \;\; P\text{-a.s. for } \; f \in b\mathcal{B}(\mathbb{R}) \text{ and } t, h \geq 0.$$

Exercise 3-2

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(B_t)_{t\geq 0}$ be a Brownian motion.

a) Show that for P-almost all ω, the path $B_\cdot(\omega)$ changes its sign infinitely many times on any interval $[0,t]$, $t \geq 0$.

b) For any $\omega \in \Omega$ we define the set

$$Z(\omega) := \big\{t \in [0,\infty) \,\big|\, B_t(\omega) = 0\big\}.$$

Show that for P-almost all ω, the set $Z(\omega)$ is closed, has Lebesgue measure 0 and has 0 as an accumulation point.

Hint: You can use Fubini for the Lebesgue measure of $Z(\omega)$.

Exercise 3-3

Given $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$, we define for any $(\mathcal{F}_t)$-stopping time τ the σ-field

$$\mathcal{F}_\tau := \big\{A \in \mathcal{F} \,\big|\, A \cap \{\tau \leq t\} \in \mathcal{F}_t \;\text{ for all } t \geq 0\big\}.$$

Let S, T be two $(\mathcal{F}_t)$-stopping times. Show that

a) if $S \leq T$, then $\mathcal{F}_S \subseteq \mathcal{F}_T$ and in general, $\mathcal{F}_{S\wedge T} = \mathcal{F}_S \cap \mathcal{F}_T$.

b) $\{S < T\}$, $\{S \leq T\}$ belong to $\mathcal{F}_S \cap \mathcal{F}_T$.
Moreover, for any $A \in \mathcal{F}_S$, $A \cap \{S < T\}$ and $A \cap \{S \leq T\}$ belong to $\mathcal{F}_{S\wedge T}$.
