ETH Zürich FS 2016
D-MATH
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# Brownian Motion and Stochastic Calculus
## Exercise Sheet 4

*Please hand in until Friday, April 15th, in your exercise group and otherwise before 13:00, in HG E 66.1*

### Exercise 4-1

Let $(B_t)_{t\geq 0}$ be a Brownian motion and define the process $(M_t)_{t\geq 0}$ by $M_t = \sup_{0\leq s\leq t} B_s$. Show that for any fixed $t \geq 0$

$$M_t - B_t \overset{Law}{=} |B_t| \overset{Law}{=} M_t.$$

**Hint:** Use (2.55) of the lecture notes.

### Exercise 4-2

Let $(X_t)_{t\geq 0}$ be the canonical one-dimensional Brownian motion and $W_0$ the Wiener measure (starting from 0). Let $S = \sup\{0 \leq u \leq 1 \,|\, X_u = 0\}$ be the time of the last zero before time 1. Show that

$$W_0\Big(S \leq s\Big) = \frac{1}{\pi}\int_0^s \frac{dv}{\sqrt{v(1-v)}} \ \left(= \frac{2}{\pi}\arcsin\sqrt{s}\right), \quad \text{for } \ 0 \leq s \leq 1.$$

**Remark:** This is the first *arcsine law* of Paul Lévy.

**Hint:** Use that $\{S \leq s\} = \{H_0 \circ \theta_s > 1 - s\}$ where $H_0 := \inf\{t \geq 0 \,|\, X_t = 0\}$ and employ the simple Markov property and the law of $H_0$ under $W_x$.

### Exercise 4-3

**a)** Prove that $P$-almost all Brownian paths are nowhere on $[0,1]$ Hölder-continuous of order $\alpha$, for any $\alpha > \frac{1}{2}$.
**Hint:** Take any $M \in \mathbb{N}$ satisfying $M(\alpha - \frac{1}{2}) > 1$ and show that the set $\{B_\cdot(\omega)$ is $\alpha$-Hölder at some $s \in [0,1]\}$ is contained in the set $\bigcup_{C\in\mathbb{N}}\bigcup_{m\in\mathbb{N}}\bigcap_{n\geq m}\bigcup_{k=0,\dots,n-1}\bigcap_{j=1}^{M}\left\{|B_{\frac{k+j}{n}}(\omega) - B_{\frac{k+j-1}{n}}(\omega)| \leq C\frac{1}{n^\alpha}\right\}$.

**b)** The *Kolmogorov-Čentsov theorem* states that a process $X$ on $[0,T]$ satisfying

$$E\big[|X_t - X_s|^\alpha\big] \leq C\,|t-s|^{1+\beta}, \quad s,t \in [0,T],$$

where $\alpha, \beta, C > 0$, has a version which is locally Hölder-continuous of order $\gamma$ for all $\gamma < \beta/\alpha$. Use this to deduce that Brownian motion has for every $\gamma < 1/2$ a version which is locally Hölder-continuous of order $\gamma$.

**Remark:** One can also show that the Brownian paths are *not* Hölder-continuous of order 1/2. The exact modulus of continuity was found by P. Lévy.

Exercise sheets and further information are also available on:

**http://www.math.ethz.ch/education/bachelor/lectures/fs2016/math/bmsc**