ETH Zürich FS 2016
D-MATH
Prof. Pierre Nolin

Coordinator
Wei Qian

Brownian Motion and Stochastic Calculus

Exercise Sheet 5

Please hand in until Friday, April 22th, in your exercise group
and otherwise before 13:00, in HG E 66.1

Exercise 5-1

Let W_x be the Wiener measure on the canonical space $(\Omega, \mathcal{F}, (\sigma(X_u, u \le t))_{t\ge 0})$ where $(X_t)_{t\ge0}$ is the canonical process. We want to show that *the one dimensional Brownian motion is recurrent.* More precisely, if we denote by A the set

$$A = \bigcap_{n\in\mathbb{N}} \{X_t = 0 \text{ for some } t \ge n\},$$

then we want to show that $W_x(A) = 1$ for all x. Let $H_0 = \inf\{s \ge 0 \mid X_s = 0\}$.

a) Show that $W_x(H_0 < \infty) \ge \frac{1}{2}$ and $W_x(A) \ge \frac{1}{2}$ for all x.
Hint: Apply the Markov property.

b) Conclude that $W_x(A) = 1$ for all x.
Hint: Use Serie 2 Exercise 3.

Exercise 5-2

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge0}, P)$ be a filtered probability space and let $(B_t)_{t\ge0}$ be an $(\mathcal{F}_t)$-adapted continuous process starting at 0. Show that the following properties are equivalent:

a) $(B_t)_{t\ge0}$ is a Brownian motion with respect to $(\mathcal{F}_t)_{t\ge0}$;
b) For all $\lambda \in \mathbb{R}$, the complex valued process $(M_t^\lambda)_{t\ge0}$ defined by $M_t^\lambda := \exp\left(i\lambda B_t + \frac{\lambda^2 t}{2}\right)$ is a martingale with respect to $(\mathcal{F}_t)_{t\ge0}$.

Exercise 5-3

Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a decreasing sequence of sub-σ-fields of $\mathcal{F}$ (i.e. $\mathcal{F}_{n+1} \subseteq \mathcal{F}_n \subseteq \mathcal{F}, \forall n \in \mathbb{N}$) and let $(X_n)_{n\in\mathbb{N}}$ be a *backward submartingale*, i.e. $E[|X_n|] < \infty$, X_n is $\mathcal{F}_n$-measurable and $E[X_n \,|\mathcal{F}_{n+1}] \ge X_{n+1}$ P-a.s. for every $n \in \mathbb{N}$.

a) We want to show that for any $n \ge m$, $N, M > 0$,

$$E\big[-X_n \mathbf{1}_{\{-X_n\ge M\}}\big] \le E\big[X_m\big] - E\big[X_n\big] + E\big[|X_m| \mathbf{1}_{\{-X_m\ge N\}}\big] + \frac{N}{M} E\big[X_n^-\big].$$

where $X_n^- = -\min\{X_n, 0\}$. We do it by two steps.

(i) Show that for any $n \ge m$, $N, M > 0$,

$$E\big[-X_n \mathbf{1}_{\{-X_n\ge M\}}\big] \le E\big[-X_n\big] - E\big[-X_m\big] + E[-X_m \mathbf{1}_{\{-X_n\ge M\}}\big]. \tag{1}$$

(ii) Show that for any $n \geq m$, $N, M > 0$,

$$E[-X_m \mathbf{1}_{\{-X_n \geq M\}}] \leq E\big[|X_m| \mathbf{1}_{\{-X_m \geq N\}}\big] + \frac{N}{M} E\big[X_n^-\big], \tag{2}$$

and conclude.
Hint: First prove that

$$E[-X_m \mathbf{1}_{\{-X_n \geq M\}}] \leq E[-X_m \mathbf{1}_{\{-X_n \geq M, -X_m \geq N\}}] + E[-X_m \mathbf{1}_{\{-X_n \geq M, 0 < -X_m < N\}}].$$

b) Assume that $\lim_{n \to \infty} E[X_n] > -\infty$, we want to show that the sequence $(X_n)_{n \in \mathbb{N}}$ is uniformly integrable. We will do it by three steps.

(i) Show that it is sufficient to show that $(X_n^-)_{n \in \mathbb{N}}$ is uniformly integrable.

(ii) Show that $\sup_{n \in \mathbb{N}} E[X_n^-] < \infty$.

(iii) Use (a) and (b ii) to conclude that $(X_n^-)_{n \in \mathbb{N}}$ is uniformly integrable.
Hint: Write

$$\sup_{n \in \mathbb{N}} E\big[-X_n \mathbf{1}_{\{-X_n \geq M\}}\big] \leq \max_{n < m} E\big[-X_n \mathbf{1}_{\{-X_n \geq M\}}\big] \vee \sup_{n \geq m} E\big[-X_n \mathbf{1}_{\{-X_n \geq M\}}\big].$$
