ETH Zürich FS 2016
D-MATH
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Brownian Motion and Stochastic Calculus
Exercise Sheet 6

Please hand in until Friday, April 29th, in your exercise group and otherwise before 13:00, in HG E 66.1

Exercise 6-1

Let $(X_t)_{t\geq 0}$ be a right-continuous submartingale adapted to the $(\mathcal{G}_t)$ filtration. Let $S \leq T$ be two bounded $(\mathcal{G}_t)$-stopping times. Show that

$$E\big[X_T \,\big|\, \mathcal{G}_S\big] \geq X_S \quad \text{a.s.}$$

Hint: Use the discrete stopping theorem which claims that the same statement holds for discrete-time submartingales. Consider the sequence $S_n(\omega) := \sum_{k=1}^{\infty} \frac{k}{2^n} \mathbf{1}_{\{\frac{k-1}{2^n} \leq S(\omega) < \frac{k}{2^n}\}}$ and define $(T_n)_{n\in\mathbb{N}}$ similarly, and then use Serie 5 Exercise 3 b) to pass to the limit.

Exercise 6-2

Let $M := (M_t)_{t\geq 0}$ be a continuous $(\mathcal{F}_t)$-martingale of finite variation. Show that

$$P\text{-a.s.}, \ \forall t \geq 0, \ M_t = M_0.$$

Hint: First, consider the case where M has uniformly bounded variation and show $E[M_t^2] = 0$ for all $t \geq 0$. Then, use a suitable stopping time T and consider the stopped process $M^T := (M_{t\wedge T})_{t\geq 0}$.

Exercise 6-3

Let $(B_t)_{t\geq 0}$ be a Brownian motion. For any $a > 0$ consider the stopping times

$$T_a := \inf\big\{t > 0 \,\big|\, B_t \geq a\big\}, \quad \overline{T}_a := \inf\big\{t > 0 \,\big|\, |B_t| \geq a\big\}$$

a) Show that $P[T_a < \infty] = 1$ and that the Laplace transform of T_a has value:

$$E\big[\exp(-\mu T_a)\big] = \exp\big(-a\sqrt{2\mu}\big), \quad \forall \mu > 0.$$

Hint: Consider the martingale $M_t^\lambda = \exp\Big(\lambda B_t - \frac{\lambda^2}{2} t\Big)$.

b) Show that the Laplace transform of $\overline{T}_a$ has value

$$E\big[\exp(-\mu \overline{T}_a)\big] = \frac{1}{\cosh(a\sqrt{2\mu})}, \quad \forall \mu > 0.$$

Hint: Consider a martingale N^λ similar to M^λ and use the same approach as in a).

Exercise sheets and further information are also available on:
http://www.math.ethz.ch/education/bachelor/lectures/fs2016/math/bmsc