ETH Zürich FS 2016
D-MATH
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Brownian Motion and Stochastic Calculus
Exercise Sheet 7

Please hand in until Friday, May 6th, in your exercise group and otherwise before 13:00, in HG E 66.1

Exercise 7-1

Let $(X_t)_{t\geq 0}$ be a Brownian motion. Show that

$$P\text{-a.s., for all } t \geq 0, \quad \int_0^t X_s \, dX_s = \frac{1}{2}\Big(X_t^2 - t\Big).$$

Hint: Fix $t \geq 0$ and use a dyadic partition of $[0,t]$ to approximate X on $[0,t]$ by a sequence of piecewise constant processes.

Exercise 7-2

Let $(B_t)_{t\geq 0}$ be a Brownian motion. Fix any $0 < T < \infty$ and let $f \in L^2([0,T])$ be a deterministic function. For any $0 \leq a < b \leq T$ we set

$$\mathcal{J}_{a,b} := \int_a^b f(s) \, dB_s.$$

a) Show that $\mathcal{J}_{a,b}$ is normally distributed with $E[\mathcal{J}_{a,b}] = 0$ and $\mathrm{Var}(\mathcal{J}_{a,b}) = \int_a^b f^2(s) \, ds$.
Hint: Assume first that f is a step function and then use Series 2 Exercise 2.

b) Let $0 \leq a < b \leq c < d \leq T$. Show that the random vector $(\mathcal{J}_{a,b}, \mathcal{J}_{c,d})$ is a Gaussian vector and that the random variables $\mathcal{J}_{a,b}$ and $\mathcal{J}_{c,d}$ are independent.
Hint: Assume again first that f is a step function and then use Series 2 Exercise 2.

Exercise 7-3

Let $(B_t)_{t\geq 0}$ be a Brownian motion. Fix any $0 < T < \infty$ and let $f \in L^2([0,T])$ be a deterministic function. For any $0 \leq a < b \leq T$ we set

$$\mathcal{J}_{a,b} := \int_a^b f(s) \, dB_s.$$

Moreover, for any $t \in [0,T]$, we denote $\mathcal{J}_t := \mathcal{J}_{0,t}$.

a) Show that the process $\mathcal{J} := (\mathcal{J}_t)_{t\in[0,T]}$ is a centered Gaussian process and calculate its covariance function.

b) Show that the process $(\mathcal{J}_t)_{t\in[0,T]}$ has the same law as the process $Y := (Y_t)_{t\in[0,T]}$ defined by

$$Y_t := B_{\int_0^t f^2(s) \, ds}.$$
