ETH Zürich FS 2016
D-MATH
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Brownian Motion and Stochastic Calculus
Exercise Sheet 8

Please hand in until Friday, May 13th, in your exercise group and otherwise before 13:00, in HG E 66.1

Exercise 8-1

Let $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, P)$ satisfying the usual conditions.

a) Let $(M_t)_{t\geq 0}$ be a right-continuous martingale and τ be any stopping time. Show that the stopped process $M^\tau := (M_{\tau\wedge t})_{t\geq 0}$ is a martingale.
Hint: Use similar arguments as in Series 6 Exercise 1.

b) Show that for a continuous local martingale $(M_t)_{t\geq 0}$ with M_0 bounded, one can find a sequence of stopping times $(S_n)_{n\in\mathbb{N}}$, P-a.s. tending to infinity, such that for each n, the stopped process $M^{S_n} := (M_{S_n\wedge t})_{t\geq 0}$ is a bounded continuous martingale.

Exercise 8-2

Let $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, P)$ satisfying the usual conditions.

a) Show that every continuous *bounded* local martingale is a martingale.

b) Let $0 < T < \infty$ be a deterministic time. Show that any nonnegative continuous local martingale $(X_t)_{t\in[0,T]}$ with $E[X_0] < \infty$ is also a supermartingale, and if

$$E[X_T] = E[X_0],$$

then $(X_t)_{t\in[0,T]}$ is a martingale.

Exercise 8-3

Let $(B_t)_{t\geq 0}$ and $(\widetilde{B}_t)_{t\geq 0}$ be two independent Brownian motions.

a) Show that $\langle B, \widetilde{B}\rangle_t = 0$ for any $t \geq 0$.
Hint: Use the *polarization formula.*

b) Consider the process $W_t = \rho B_t + \sqrt{1-\rho^2}\widetilde{B}_t$, with $\rho \in [-1, 1]$. Show that $(W_t)_{t\geq 0}$ is a Brownian motion and compute $\langle W, B\rangle_t$ for any $t \geq 0$.
