ETH Zürich FS 2016
D-MATH
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Brownian Motion and Stochastic Calculus
Exercise Sheet 9

Please hand in until Friday, May 20th, in your exercise group
and otherwise before 13:00, in HG E 66.1

Exercise 9-1

Show that when $(M_t)_{t\geq 0}$ is a continuous local martingale such that $M_0 = 0$ and $E[\langle M\rangle_t] < \infty$ for all $t \geq 0$, then $(M_t)_{t\geq 0}$ is a continuous square integrable martingale.

Exercise 9-2

For continuous semimartingales X and Y, the *Stratonovich integral* is defined by

$$\int Y \circ dX := \int Y\, dX + \frac{1}{2}\langle Y, X\rangle.$$

a) Show that for suitable functions F (which ones?) the chain rule for the Stratonovich integral holds, i.e. show that

$$F(X_t) = F(X_0) + \int_0^t F'(X_s) \circ dX_s.$$

Show that if X is a local martingale, then $\int Y \circ dX$ is in general not a local martingale.

b) Show that there is a sequence $(\Pi_n)_{n\in\mathbb{N}}$ of partitions of $[0,\infty)$ such that with probability one,

$$\int_0^t Y_s \circ dX_s = \lim_{n\to\infty} \sum_{t_i\in\Pi_n} \frac{1}{2}\big(Y_{t_{i+1}\wedge t} + Y_{t_i\wedge t}\big)\big(X_{t_{i+1}\wedge t} - X_{t_i\wedge t}\big)$$

simultaneously for all $t \geq 0$.
Hint: Use the fact that for continuous semimartingales X,Y, we can find a sequence of partitions $(\Pi_n)_{n\in\mathbb{N}}$ such that $\mathbb{P} - a.s.$

$$\int_0^t Y_s\, dX_s = \lim_{n\to\infty} \sum_{t_i\in\Pi_n} (Y_{t_i\wedge t})\,(X_{t_{i+1}\wedge t} - X_{t_i\wedge t})$$

simultaneously for all $t \geq 0$.

Please turn the page

Exercise 9-3

Let B be a Brownian motion in $\mathbb{R}^3$, $0 \neq x \in \mathbb{R}^3$ and define the process $M = (M_t)_{t \geq 0}$ by

$$M_t = \frac{1}{|x + B_t|}.$$

This is well defined since one can show that $P[B_t = -x \text{ for some } t \geq 0] = 0$.

a) Show that M is a continuous local martingale.
Hint: Use Itô's formula.

Moreover, show that M is bounded in L^2, i.e., $\sup_{t \geq 0} E[|M_t|^2] < \infty$.
Hint: For any $t \geq 0$, show that

$$E\left[|M_t|^2 1_{\{|M_t| \geq \frac{2}{|x|}\}}\right] = (2\pi t)^{-\frac{3}{2}} \int\limits_{|y| \leq \frac{|x|}{2}} \frac{1}{|y|^2} \exp\left(-\frac{|y-x|^2}{2t}\right) dy$$

and estimate the right-hand side from above using the reverse triangle inequality.

b) Show that M is a *strict local martingale*, i.e., M is not a martingale.
Hint: Show that $E[M_t] \to 0$ as $t \to \infty$. To this end, similarly to part a), compute $E[M_t]$ and use the reverse triangle inequality as a first estimate. Then compute the resulting integral using spherical coordinates.
Remark: This is the standard example of a local martingale which is not a (true) martingale. It also shows that even good integrability properties like boundedness in L^2 are not enough to guarantee the martingale property.
