ETH Zürich FS 2016
D-MATH
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Brownian Motion and Stochastic Calculus
Solution 1

Solution 1-1

a) We show that $Z \sim \mathcal{N}(0,1)$ by calculating its characteristic function. Using the independence of X and Y and that X and $-X \sim \mathcal{N}(0,1)$, we get for each $t \in \mathbb{R}$ that

$$\begin{aligned}\varphi_Z(t) := E\big[e^{itZ}\big] &= E\big[e^{itX}\,1_{\{Y=1\}}\big] + E\big[e^{-itX}\,1_{\{Y=-1\}}\big] \\ &= E\big[e^{itX}\big]\;P\big[Y=1\big] + E\big[e^{-itX}\big]\;P\big[Y=-1\big] \\ &= e^{-\frac{1}{2}t^2}.\end{aligned}$$

To prove that (X,Z) is a Gaussian vector, we need to show that for any $\lambda_1, \lambda_2 \in \mathbb{R}$, the random variable $\lambda_1 X + \lambda_2 Z$ is normal distributed. Fix any $\lambda_1, \lambda_2 \in \mathbb{R}$.
For $p \in \{0,1\}$ we see that

$$\lambda_1 X + \lambda_2 Z = cX$$

for some $c \in \mathbb{R}$. Therefore, as $X \sim \mathcal{N}(0,1)$ we get that $\lambda_1 X + \lambda_2 Z \sim \mathcal{N}(0,c^2)$ and thus (X,Z) is a Gaussian vector.
Now, let $p \in [0,1] \setminus \{0,1\}$. Assume by contradiction that (X,Z) is a Gaussian vector. Then $X+Z$ is normal distributed. But since $P[X=0]=0$ as X is normal distributed, we get that

$$P\big[X+Z=0\big] = P\big[Y=-1\big] = 1-p \neq 0$$

which gives a contradiction. We conclude that

$$(X,Z) \text{ is a Gaussian vector } \iff p \in \{0,1\}.$$

b) Using that $X \sim \mathcal{N}(0,1)$, the independence of X and Y and that $E[Y] = 2p-1$, we get

$$\begin{aligned}\mathrm{Cov}\big(X,Z\big) &= E\big[X^2\,Y\big] - E\big[X\big]\;E\big[X\,Y\big] \\ &= E\big[X^2\big]\;E\big[Y\big] \\ &= \mathrm{Var}\big(X\big)\;E\big[Y\big] \\ &= 2p-1.\end{aligned}$$

Therefore,

$$\mathrm{Cov}\big(X,Z\big) = 0 \iff p = 1/2.$$

c) Assume by contradiction that X and Z are independent. Then, as $Z \sim \mathcal{N}(0,1)$,

$$0 = P\Big[|Z|>1\,\Big|\,|X|\le 1\Big] = P\Big[|Z|>1\Big] \neq 0$$

which gives a contradiction.
Alternative proof: For $p \in (0,1)$, if X and Z were independent, since by a) X and Z are normal distributed, (X,Z) would be a Gaussian vector, which is a contradiction to a). For $p \in \{0,1\}$, it is clear that we do not have independence, since in that case

$$X = Z \text{ a.s.} \quad \text{or} \quad X = -Z \text{ a.s.}$$

Remark: This exercise shows that a random vector, which each component is normal distributed does **not** need to be a Gaussian vector. Moreover, the exercise shows that two uncorrelated random variables do **not** need to be independent.

Solution 1-2

a) We need to show that for any $n \geq 1$ and any $0 \leq t_1 \leq t_2 ... \leq t_n \leq 1$ the random vector $(X_{t_1}, ..., X_{t_n})$ is a Gaussian vector. However any linear combination of $X_{t_1}, ..., X_{t_n}$ is also a linear combination of $W_{t_1}, ..., W_{t_n}, W_1$ which is Gaussian because the Brownian motion is a Gaussian process.

For any $t \in [0, 1]$ we have

$$E[X_t] = E[W_t - t\,W_1] = 0.$$

For any $0 \leq s, t \leq 1$, using that $\mathrm{Cov}(W_t, W_s) = t \wedge s$ (see Prop 1.15), we have

$$\begin{aligned}\mathrm{Cov}(X_t, X_s) &= \mathrm{Cov}(W_t - t\,W_1, W_s - s\,W_1) \\ &= \mathrm{Cov}(W_t, W_s) - s\,\mathrm{Cov}(W_t, W_1) - t\,\mathrm{Cov}(W_1, W_s) + ts\,\mathrm{Cov}(W_1, W_1) \\ &= t \wedge s - ts.\end{aligned}$$

b) Take any $t \in (0, 1)$. We show that the increment $X_1 - X_t$, $X_t - X_0$ are correlated. In the same way as above we obtain that

$$\mathrm{Cov}(X_1 - X_t,\, X_t - X_0) = \mathrm{Cov}(-W_t + t\,W_1,\, W_t - t\,W_1) = t(t-1) \neq 0.$$

Solution 1-3

a) We just show that the fact that X is a modification of Y implies the indistinguishability, since the converse is obvious. Without loss of generality, we assume that X and Y are right-continuous.

For $t \geq 0$, we define the null set $N_t := \{\omega : X_t(\omega) \neq Y_t(\omega)\}$. We consider $N := \cup_{t \in \mathbb{Q}_+} N_t$, which remains a null set as a countable union of null sets. Finally, we introduce the null set $A_Z := \{\omega : Z_\cdot(\omega) \text{ not right-continuous}\}$ for $Z = X, Y$ and we define $M := A_X \cup A_Y \cup N$, which is still a null set.

It suffices to check that, for all $\omega \in M^c$, $X_t(\omega) = Y_t(\omega)\ \forall\, t \geq 0$. By definition of M we clearly have that, for $\omega \in M^c$, $X_t(\omega) = Y_t(\omega)\ \forall\, t \in \mathbb{Q}_+$. Now, take any $t \geq 0$ and let (t_n) be a sequence in $\mathbb{Q}_+$ with $t_n \downarrow t$. The right-continuity of the paths $X_\cdot(\omega)$ and $Y_\cdot(\omega)$ then implies $X_t(\omega) = \lim_{n\to\infty} X_{t_n}(\omega) = \lim_{n\to\infty} Y_{t_n}(\omega) = Y_t(\omega)$.

b) Take $\Omega = [0, \infty)$, $\mathcal{F} = \mathcal{B}([0, \infty))$ the Borel σ-algebra, and P a probability measure with $P(\{\omega\}) = 0\ \forall\, \omega \in \Omega$ (for instance, the exponential distribution).

Set $X \equiv 0$ and $Y_t(\omega) = \begin{cases} 1, & t = \omega, \\ 0, & \text{else.} \end{cases}$

Then, $P[X_t = Y_t] = 1\ \forall\, t \geq 0$, since single points have no mass, but $\{X_t = Y_t\ \forall\, t \geq 0\} = \emptyset$. Note that all sample paths of X are continuous, while all sample paths of Y are discontinuous at $t = \omega$.