ETH Zürich FS 2016
D-MATH
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Brownian Motion and Stochastic Calculus
Solution 10

Solution 10-1

a) The function sign$(\cdot)$ is a bounded function and so the stochastic integral is well defined and a continuous local martingale. Its quadratic variation is

$$\langle X \rangle_t = \int_0^t \operatorname{sign}^2(B_s)\, ds = \int_0^t ds = t,$$

and hence by Levy's characterization theorem, we see that $(X_t)_{t\geq 0}$ is a Brownian motion. For the second part, by applying the Itô isometry, we obtain that:

$$E\big[X_t\, B_s\big] = E\Big[\int_0^s dB_u \int_0^t \operatorname{sign}(B_u) dB_u\Big] = E\Big[\int_0^{\min(s,t)} \operatorname{sign}(B_u)\, du\Big].$$

By Fubini, we get that

$$E\Big[\int_0^{\min(s,t)} \operatorname{sign}(B_u)\, du\Big] = \int_0^{\min(s,t)} E\big[\operatorname{sign}(B_u)\big]\, du = 0.$$

Where the last equation follows from the symmetry of the Brownian motion, i.e.

$$E\big[\operatorname{sign}(B_u)\big] = P\big[B_u \geq 0\big] - P\big[B_u < 0\big] = 0.$$

b) By Itô's Formula, we know that $B_t^2 = 2\int_0^t B_s\, dB_s + t$. Since $E[X_t] = 0$, we conclude using Itô's isometry and Fubini that

$$\begin{aligned} E\big[B_t^2 X_t\big] &= E\Big[\Big(\int_0^t \operatorname{sign}(B_s)\, dB_s\Big)\Big(2\int_0^t B_s\, dB_s + t\Big)\Big] \\ &= 2E\Big[\int_0^t \operatorname{sign}(B_s) B_s\, ds\Big] \\ &= 2\int_0^t E\big[|B_s|\big]\, ds. \end{aligned}$$

Finally, as $B_s \sim \mathcal{N}(0, s)$, we obtain that $E[|B_s|] = \frac{\sqrt{2s}}{\sqrt{\pi}}$. Therefore,

$$E\big[B_t^2 X_t\big] = 2\int_0^t \frac{\sqrt{2s}}{\sqrt{\pi}}\, ds = 2^{\frac{5}{2}} t^{\frac{3}{2}} \frac{1}{3\sqrt{\pi}}.$$

Therefore, as $E[X_t] = 0$, X and B cannot be independent.

Solution 10-2

a) Let X be the canonical Brownian motion in $\mathbb{R}^d, d \geq 2$, starting from $x \neq 0$. By Itô's formula, we have for any $f \in C^2$ that

$$f(X_t) = f(X_0) + \int_0^t \nabla f(X_s)\, dX_s + \frac{1}{2}\int_0^t \triangle f(X_s)\, ds.$$

Taking $f(x) := |x|$ on $\mathbb{R}^d \setminus \{0\}$, we obtain that $\frac{\partial f}{\partial x_i} = \frac{x_i}{|x|}$ and $\triangle f = \frac{d-1}{|x|}$. Therefore, we get W_x-a.s. for all $t \geq 0$,

$$R_t := |X_t| = |x| + \sum_{i=1}^d \int_0^t \frac{X_s^i}{|X_s|}\, dX_s^i + \frac{d-1}{2}\int_0^t \frac{1}{|X_s|}\, ds.$$

b) Define the process $B := (B_t)_{t\geq 0}$ by

$$B_t := \sum_{i=1}^d \int_0^t \frac{X_s^i}{|X_s|}\, dX_s^i.$$

By construction, B is a continuous local martingale. Moreover, as $\langle X^i, X^j\rangle_t = \delta_{ij}t$, we get that for any $t \geq 0$,

$$\langle B\rangle_t = \sum_{i=1}^d \int_0^t \frac{(X_s^i)^2}{|X_s|^2}\, ds = t.$$

Therefore, we conclude by Lévy's characterization theorem that W is a Brownian motion. Moreover, by a), we get that

$$R_t = |X_t| = |x| + B_t + \frac{d-1}{2}\int_0^t \frac{1}{R_s}\, ds.$$

Solution 10-3

a) By the Theorem 7.1 of the lecture notes, we get directly the existence and uniqueness of a strong solution of the SDE. So we just need to check that X_t^x is indeed a solution. Define the process $Y_t := x + \sigma\int_0^t e^{\gamma s} dB_s$, so that $X_t^x = f(t, Y_t)$, with $f(t,y) = e^{-\gamma t}y$. The derivatives of f are

$$\begin{aligned}\frac{\partial f}{\partial t} &= -\gamma f\\ \frac{\partial f}{\partial y} &= e^{-\gamma t}\\ \frac{\partial^2 f}{\partial y^2} &= 0.\end{aligned}$$

Applying Itô's formula to $f(t, Y_t)$ we get

$$\begin{aligned}df(t,Y_t) &= \frac{\partial f(t,Y_t)}{\partial t}dt + \frac{\partial f(t,Y_t)}{\partial y}dY_t + \frac{1}{2}\frac{\partial^2 f(t,Y_t)}{\partial y^2}d\langle Y\rangle_t\\ &= -\gamma f(t,Y_t)dt + e^{-\gamma t}dY_t = -\gamma X_t^x dt + e^{-\gamma t}dY_t.\end{aligned}$$

Now

$$e^{-\gamma t} dY_t = e^{-\gamma t} \sigma e^{\gamma t} dB_t = \sigma dB_t,$$

so

$$dX_t^x = df(t, Y_t) = -\gamma X_t^x dt + \sigma dB_t.$$

Finally, $X_0^x = x$, so we are done.

b) From Series 7 Exercise 2, we know that for a deterministic function $f \in L^2([0,t])$, the stochastic integral $\mathcal{J}_{0,t} := \int_0^t f(s)\, dB_s$ is normally distributed with $E[\mathcal{J}_{0,t}] = 0$ and $\mathrm{Var}(\mathcal{J}_{0,t}) = \int_0^t f^2(s)\, ds$. Therefore, for $f(s) := e^{-\gamma t}\, \sigma\, e^{\gamma s} \in L^2([0,t])$, we conclude that X_t^x is normally distributed with

$$\begin{aligned} E[X_t^x] &= e^{-\gamma t} x + E[\mathcal{J}_{0,t}] = e^{-\gamma t} x, \\ \mathrm{Var}(X_t^x) &= \mathrm{Var}(\mathcal{J}_{0,t}) = \int_0^t \left(e^{-\gamma t}\, \sigma\, e^{\gamma s}\right)^2 ds = \frac{\sigma^2}{2\gamma}\left(1 - e^{-2\gamma t}\right). \end{aligned}$$