ETH Zürich FS 2016
D-MATH
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Brownian Motion and Stochastic Calculus
Solution 11

Solution 11-1

a) We see that the SDE is of the form

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t, \quad X_0 = x \in \mathbb{R},$$

where

$$b(x) = \sqrt{1+x^2} + \frac{x}{2} \quad \text{and} \quad \sigma(x) = \sqrt{1+x^2}.$$

We observe that

$$\sup_{x\in\mathbb{R}} |\sigma'(x)| = \sup_{x\in\mathbb{R}} \left|\frac{x}{\sqrt{1+x^2}}\right| \leq 1$$

as well as

$$\sup_{x\in\mathbb{R}} |b'(x)| = \sup_{x\in\mathbb{R}} \left|\frac{x}{\sqrt{1+x^2}} + \frac{1}{2}\right| \leq \frac{3}{2}.$$

Thus, we obtain for $K := \frac{5}{2}$ that $b(\cdot)$ and $\sigma(\cdot)$ satisfy the Lipschitz condition:

$$|b(y) - b(z)| + |\sigma(y) - \sigma(z)| \leq K|y - z|, \quad y, z \in \mathbb{R}.$$

Thus we get for any $x \in \mathbb{R}$ the existence of a unique strong solution directly from Theorem 7.1 in the lecture notes.

b) We consider the function $f(x) := \operatorname{arsinh}(x) \in C^2$ (i.e. the inverse function of the hyperbolic sine). Thus, we obtain that

$$f'(x) = \frac{1}{\sqrt{1+x^2}} \quad \text{and} \quad f''(x) = -\frac{x}{(1+x^2)^{3/2}}.$$

Thus, applying Itô's formula to $Y_t := f(X_t)$, we obtain that

$$dY_t = df(X_t) = dt + dB_t, \quad Y_0 = \operatorname{arsinh}(x).$$

which implies that

$$X_t = \sinh(Y_t) = \sinh\big(\operatorname{arsinh}(x) + t + B_t\big), \quad t \geq 0.$$

Solution 11-2

a) Applying Itô's formula, we obtain that P-a.s., for all $t \geq 0$:

$$Y_t = \phi(X_t) = \phi(x) + \int_0^t \phi'(X_s)g(X_s)\,dB_s + \int_0^t \Big[\phi'(X_s)f(X_s) + \frac{1}{2}\phi''(X_s)g^2(X_s)\Big]\,ds.$$

Thus, we obtain that Y is a local martingale if for any $x \in \mathbb{R}$, $\phi(x)$ satisfies the following ordinary differential equation

$$\phi'(x)f(x) + \frac{1}{2}\phi''(x)g^2(x) = 0.$$

It is easy to check that the general solution of the above ordinary differential is of the form

$$\phi(x) = a + b\int_0^x \exp\Big(-2\int_0^u \frac{f(v)}{g^2(v)}\,dv\Big)\,du, \quad a, b \in \mathbb{R}. \tag{1}$$

For the second part, let $\phi(x)$ be of the form (1) with $b \neq 0$ (i.e. a non trivial solution). We first observe that ϕ is continuous and strictly increasing hence the inverse function of ϕ, denoted by ϕ^{-1}, exists. From previous computations, we know that P-a.s. for any $t \geq 0$

$$Y_t = \phi(X_t) = \phi(x) + \int_0^t \phi'(X_s)g(X_s)\,dB_s. \tag{2}$$

Thus, as $X_t = \phi^{-1}(Y_t)$, we get that Y_t satisfies the SDE

$$dY_t = (\phi' \circ \phi^{-1})(Y_t)\,(g \circ \phi^{-1})(Y_t)\,dB_t, \quad Y_0 = \phi(x).$$

b) As $\phi(X_t)$ is a continuous local martingale of the form (2), by Series 9 Exercise 1, it is enough to check that for any $T > 0$

$$E\Big[\int_0^T \big(\phi'(X_s)g(X_s)\big)^2\,ds\Big] < \infty$$

to conclude that $(\phi(X_t))_{t\geq 0}$ is a true martingale. First, we observe that due to our additional assumption on f being negative on $(-\infty, 0]$ and positive on $(0, \infty)$, we obtain that

$$\sup_{x\in\mathbb{R}} \big|\phi'(x)\big| \leq |b|.$$

Moreover, as $g : \mathbb{R} \to \mathbb{R}$ is Lipschitz-continuous, there exists a constant $k > 0$ such that

$$|g(x)| \leq |g(0)| + k|x|.$$

As for any $a, b \in \mathbb{R}$, we have $(a + b)^2 \leq 2(a^2 + b^2)$, we obtain that

$$g(x)^2 \leq 2g(0)^2 + 2k^2\,x^2$$

We conclude that there are constants $C, D > 0$ such that

$$E\Big[\int_0^T \big(\phi'(X_s)g(X_s)\big)^2\,ds\Big] \leq C + D\,E\Big[\int_0^T X_s^2\,ds\Big].$$

But this is finite as $(X_t)_{t\geq 0}$ is by assumption the strong solution of the SDE

$$dX_t = f(X_t)\,dt + g(X_t)\,dB_t, \quad X_0 = x.$$

and therefore, for any $T > 0$

$$E\Big[\big(\sup_{0\leq t\leq T} |X_t|\big)^2\Big] = E\Big[\sup_{0\leq t\leq T} |X_t|^2\Big] < \infty.$$

Solution 11-3

We see that the SDE is of the form

$$dX_t = a(t, X_t)\, dt + b(t, X_t)\, dB_t, \quad X_0 = \xi$$

where $a(t, x) = r_t x$ and $b(t, x) = v_t x$.

There is a theorem that for this kind of SDE, a sufficient condition for the existence of strong solution is

1. There exist $K > 0$ such that

$$\big|a(t,x) - a(t,y)\big| + \big|b(t,x) - b(t,y)\big| \le K|x-y|$$

2. There exist $K > 0$ such that

$$\big|a(t,x)\big| + \big|b(t,x)\big| \le K\big(1 + |x|\big).$$

By the boundedness of the functions r and v we get that

$$K := 2\max\Big\{ \sup_{t\ge 0} |r_t|, \sup_{t\ge 0} |v_t| \Big\} < \infty.$$

Moreover, we see directly that for any $t \in [0, \infty)$ and $x, y \in \mathbb{R}$

$$\big|a(t,x) - a(t,y)\big| + \big|b(t,x) - b(t,y)\big| \le K|x-y|$$

as well as

$$\big|a(t,x)\big| + \big|b(t,x)\big| \le K\big(1 + |x|\big).$$

Thus, as $\xi \in L^2(\Omega, \mathcal{F}_0, P)$, we get that the SDE has a unique strong solution.
For the second part of the exercise, namely to get the explicit expression for the unique strong solution of the SDE, we first *guess* the solution and then we verify that our guess satisfy indeed the SDE. For our guess, we apply *purely formally* Itô's formula to $Y := \log(X)$. It is *only formally and not mathematically correct*, since we don't know yet that the solution of the SDE X will take values only in $[0, \infty)$ and thus we cannot apply the logarithm on X! Still, by formally applying Itô's formula to $Y := \log(X)$ we obtain that

$$dY_t = \frac{1}{X_t}\, dX_t - \frac{1}{2X_t^2}\, d\langle X\rangle_t = r_t\, dt + v_t\, dB_t - \frac{v_t^2}{2}\, dt$$

Thus, as $Y := \log(X)$ and $X_0 = \xi$, our *guess* of the solution of the SDE is

$$X_t = \xi \, \exp\bigg(\int_0^t \Big(r_s - \frac{v_s^2}{2} \Big)\, ds + \int_0^t v_s\, dB_s \bigg).$$

We verify that our guess satisfies $X_0 = \xi$ and, by applying Itô's formlula, solves the SDE. Thus we have found the unique solution of the SDE.