Brownian Motion and Stochastic Calculus
Solution 12

Solution 12-1

We consider for any fixed $0 \leq \Theta \leq \infty$ the process $(S_t^{(\Theta)})_{t\geq 0}$

$$S_t^{(\Theta)} = \begin{cases} 0, & 0 \leq t < \beta_\Theta, \\ B_t, & \beta_\Theta \leq t < \infty \end{cases}$$

where $\beta_\Theta := \inf \{ s \geq \Theta \,|\, B_s = 0 \}$. We observe that β_Θ is a stopping time (w.r.t. any filtration satisfying the usual conditions such that B is adapted to). As a consequence of Series 8 Exercise 1, we obtain that for any semimartingale $(S_t)_{t\geq 0}$ and any stopping time τ the stopped process $(S_t^\tau)_{t\geq 0}$ is also a semimartingale. As obviously the difference of two semimartingales is again a semimartingale, we see, as $B_{\beta_\Theta} = 0$, that

$$S^{(\Theta)} = B - B^{\beta_\Theta}$$

is a continuous semimartingale. Moreover, we have for any $t \geq 0$ that $X_t^{(\Theta)} = f(S_t^{(\Theta)})$ for the C^2 function $f(x) := x^3$. Thus, by applying Itô's formula, we get that

$$\begin{aligned} X_t^{(\Theta)} = f\left(S_t^{(\Theta)}\right) &= \left(S_0^{(\Theta)}\right)^3 + \int_0^t 3\left(S_s^{(\Theta)}\right)^2 dS_s^{(\Theta)} + 3\int_0^t S_s^{(\Theta)}\, d\langle S^{(\Theta)}\rangle_s \\ &= \int_0^t 3\left(X_s^{(\Theta)}\right)^{2/3} dS_s^{(\Theta)} + 3\int_0^t \left(X_s^{(\Theta)}\right)^{1/3} d\left\langle S^{(\Theta)}\right\rangle_s \\ &= \int_0^t 3\left(X_s^{(\Theta)}\right)^{2/3} \mathbf{1}_{s>\beta_\Theta}\, dB_s + 3\int_0^t \left(X_s^{(\Theta)}\right)^{1/3} \mathbf{1}_{s>\beta_\Theta}\, ds \\ &= \int_0^t 3\left(X_s^{(\Theta)}\right)^{2/3} dB_s + 3\int_0^t \left(X_s^{(\Theta)}\right)^{1/3} ds \end{aligned}$$

We conclude that $X^{(\Theta)}$ solves the SDE for each $0 \leq \Theta \leq \infty$.

Solution 12-2

a) In order to find the SDEs for $\widetilde{X}$ and $\widetilde{S}$ we can use Itô's formula.
Consider the function $f(x,y) = \frac{x}{y} \in C^\infty(\mathbb{R} \times (0,\infty))$, then

$$\frac{\partial f}{\partial x} = \frac{1}{y} \qquad \frac{\partial f}{\partial y} = -\frac{x}{y^2}$$
$$\frac{\partial^2 f}{\partial x^2} = 0 \qquad \frac{\partial^2 f}{\partial y^2} = \frac{2x}{y^3} \qquad \frac{\partial^2 f}{\partial x \partial y} = -\frac{1}{y^2}.$$

Applying Itô's formula

$$d\widetilde{S} = df(S,X) = \frac{1}{X_t}dS_t - \frac{S_t}{\left(X_t\right)^2}dX_t + \frac{1}{2}\frac{2S_t}{X_t^3}d\left\langle X\right\rangle_t - \frac{1}{X_t^2}d\left\langle S,X\right\rangle_t, \tag{1}$$

where

$$\left\langle X\right\rangle_t = \left\langle \int_0^\cdot aX_u dB_u \right\rangle_t = \int_0^t a^2 X_u^2 du \tag{2}$$

$$\left\langle S,X\right\rangle_t = \left\langle \int_0^\cdot \sigma S_u dW_u, \int_0^\cdot aX_u dB_u \right\rangle_t = \int_0^t a\sigma X_u S_u d\left\langle W,B\right\rangle_u = 0 \tag{3}$$

Replacing (2) and (3) in (1) we obtain

$$d\widetilde{S}_t = \widetilde{S}_t\Big[(\mu - b + a^2)dt + \sigma dW_t - adB_t\Big]. \tag{4}$$

Since S and X satisfy exactly the same type of differential equations just with different letters. The SDE of $\tilde{S}$ is obtained from (4) by exchanging $\tilde{S}$ and $\tilde{X}$, σ and a, b and μ, W and B. We get

$$d\widetilde{X} = \widetilde{X}_t\Big[(\sigma^2 - \mu + b)dt - \sigma dW_t + adB_t\Big]. \tag{5}$$

b) For both SDEs we can find explicitly the solution. From (4) it is clear that $\widetilde{S}$ is the stochastic exponential of

$$L_t^{(1)} = (\mu - b + a^2)t + \sigma W_t - aB_t$$

while, from (5), $\widetilde{X}$ is the stochastic exponential

$$L_t^{(2)} = (\sigma^2 - \mu + b)t - \sigma W_t + aB_t.$$

These two processes are continuous semimartingales. Although the notion of stochastic exponential was only introduced for local martingales in the lectures, it actually also applies for semimartingales. Note that (4), (5) clearly satisfy the Lipschitz condition and thus have a unique strong solution. It is not hard to verify that $\widetilde{S}$ and $\widetilde{X}$ are given by

$$\widetilde{S}_t = \widetilde{S}_0 \exp\left(L_t^{(1)} - \frac{1}{2}\left\langle L^{(1)}\right\rangle_t\right) = \widetilde{S}_0 \exp\left((\mu - b + a^2)t + \sigma W_t - aB_t - \frac{1}{2}\sigma^2 t - \frac{1}{2}a^2 t\right)$$

$$\widetilde{X}_t = \widetilde{X}_0 \exp\left(L_t^{(2)} - \frac{1}{2}\left\langle L^{(2)}\right\rangle_t\right) = \widetilde{X}_0 \exp\left((\sigma^2 + b - \mu)t - \sigma W_t + aB_t - \frac{1}{2}\sigma^2 t - \frac{1}{2}a^2 t\right)$$

Having the explicit solution, it is clear that

- $\widetilde{S}$ is sub-/super-/ martingale if $\mu - b + a^2 \geq 0$ / $\mu - b + a^2 \leq 0$ / $\mu - b + a^2 = 0$.
- $\widetilde{X}$ is sub-/super-/ martingale if $-\mu + b + \sigma^2 \geq 0$ / $-\mu + b + \sigma^2 \leq 0$ / $-\mu + b + \sigma^2 = 0$.

Solution 12-3

Let P_{x_0} be the unique solution of the martingale problem (L, x_0) on $\big(C(\mathbb{R}_+, \mathbb{R}), \mathcal{F}\big)$ as in (7.2) in the lecture notes, but with $L := \frac{1}{2}\frac{\partial^2}{\partial x^2} + \big(1 + b(x)\big)\frac{\partial}{\partial x}$. We are first going to characterize P_{x_0} explicitly. Fix any $f \in C^2$ and consider

$$M_t^f := f(X_t) - f(X_0) - \int_0^t Lf(X_s)\,ds, \quad t \geq 0.$$

Applying Itô's formula on $f(X_t)$ and using the definition of L above, we obtain under W_{x_0} that

$$M_t^f = \int_0^t \frac{\partial}{\partial x} f(X_s)\,dX_s - \int_0^t \big(1 + b(X_s)\big)\frac{\partial}{\partial x} f(X_s)\,ds \quad W_{x_0}\text{-a.s.} \tag{6}$$

We define the W_{x_0}-local martingale

$$Z_t := \exp\bigg(\int_0^t 1 + b(X_s)\,dX_s - \frac{1}{2}\int_0^t \big(1 + b(X_s)\big)^2\,ds\bigg), \quad t \ge 0$$

We denote by $E_{x_0}[\cdot]$ the expectation with respect to W_{x_0}. Since by assumption $b(\cdot)$ is bounded, we obtain for each $t \ge 0$ that

$$E_{x_0}\bigg[\exp\Big(\frac{1}{2}\int_0^t \big(1 + b(X_s)\big)^2\,ds\Big)\bigg] < \infty.$$

We conclude from Novikov's condition that Z is a true martingale. Thus, for each fixed $T > 0$, we can introduce a new probability measure on $\big(C(\mathbb{R}_+, \mathbb{R}), \mathcal{F}\big)$ defined by

$$Q_T := Z_T\,W_{x_0}.$$

We see that for any $A \in \mathcal{F}_T$, we have $Q_T[A] = E_{x_0}[Z_T\,\mathbf{1}_A]$, in particular Q_T is absolutely continuous with respect to W_{x_0} on $\mathcal{F}_T$. Moreover, by Girsanov transformation, we obtain that $B_t := X_t - \int_0^t \big(1 + b(X_s)\big)\,ds, \;\; 0 \le t \le T$, is a Q_T-local martingale. We deduce from (6) that $(M_t^f)_{0\le t\le T}$ is a Q_T-local martingale and hence by uniqueness we get $Q_T|_{\mathcal{F}_T} = P_{x_0}|_{\mathcal{F}_T}$. So we have proved the first part.
For the second part, observe that due to Lévy's characterization, the process $(B_t)_{0\le t\le T}$ is a P_{x_0}- Brownian motion. Since $T > 0$ was arbitrarily chosen, we obtain that $(B_t)_{0\le t<\infty}$ is a P_{x_0}-Brownian motion. We define the set

$$A := \Big\{\omega \,\Big|\, \limsup_{t\to\infty} \frac{X_t(\omega)}{t} = 0\Big\}.$$

As X is a Brownian motion under W_{x_0}, we obtain by the law of iterated logarithm that $W_{x_0}(A) = 1$. But by definition of B we have for any ω

$$\frac{X_t(\omega)}{t} = \frac{B_t(\omega)}{t} + \frac{1}{t}\int_0^t \big(1 + b(X_s(\omega))\big)\,ds.$$

As $b(\cdot) \ge 0$ and B is a P_{x_0}-Brownian motion, we deduce from the law of iterated logarithm applied on B that $\liminf_{t\to\infty} \frac{X_t}{t} \ge 1$ P_{x_0}-a.s. Thus $P_{x_0}(A) = 0$ or equivalently $P_{x_0}(A^c) = 1$.

Solution 12-4

We argue similarly as on page 114-115 of the lecture notes. For $m \geq 1$ large enough so that $\frac{1}{m} < d(x, U^c)$, we define $T_m := \inf\big\{s \geq 0\,;\, d(X_s, U^c) \leq \frac{1}{m}\big\}$ and construct $u_m \in C_c^2(\mathbb{R}^d, \mathbb{R})$ such that $u = u_m$ on $\big\{z \in U\,;\, d(z, U^c) \geq \frac{1}{m}\big\}$. We apply Itô's formula to $u_m(X_t)\, \exp\big(\int_0^t c(X_s)\, ds\big)$, take then the expectation and use (7.38) to obtain that

$$\begin{aligned} &E\Big[u_m(X_{t\wedge T_m})\, \exp\Big(\int_0^{t\wedge T_m} c(X_s)\, ds\Big)\Big] - u_m(x) \\ &= E\Big[\int_0^{t\wedge T_m} \big[Lu_m(X_s) + c(X_s)u_m(X_s)\big]\, \exp\Big(\int_0^s c(X_r)\, dr\Big)\, ds\Big]. \end{aligned}$$

Now, as $u_m = u$ on $\big\{z \in U\,;\, d(z, U^c) \geq \frac{1}{m}\big\}$, by definition of T_m and as u is the solution of the boundary problem, we obtain that

$$u(x) = E\Big[u(X_{t\wedge T_m})\, \exp\Big(\int_0^{t\wedge T_m} c(X_s)\, ds\Big)\Big] + E\Big[\int_0^{t\wedge T_m} \Big[f(X_s)\, \exp\Big(\int_0^s c(X_r)\, dr\Big)\Big]\, ds\Big]$$

Since $T_m \uparrow T_U < \infty$ due to (7.31), we can let $t \to \infty$ and then $m \to \infty$ to conclude, by dominated convergence theorem, that

$$u(x) = E\Big[g(X_{T_U})\, \exp\Big(\int_0^{T_U} c(X_s)\, ds\Big)\Big] + E\Big[\int_0^{T_U} f(X_s)\, \exp\Big(\int_0^s c(X_r)\, dr\Big)\, ds\Big].$$