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D-MATH
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Brownian Motion and Stochastic Calculus
Solution 2

Solution 2-1

a) Fix any $t \in \mathbb{R}$. Since Brownian motion B is a Gaussian process, we get by definition that X_t is Gaussian distributed. It remains to check its mean and variance:

$$\begin{aligned} \mathrm{E}[X_t] &= 0, \\ \mathrm{Var}(X_t) &= e^{-2t}e^{2t} = 1. \end{aligned}$$

b) Fix any $n \in \mathbb{N}$ and any $t_1, t_2, ..., t_n \geq 0$. It is enough to check that

$$\big(X_{-t_1}, X_{-t_2}, ..., X_{-t_n}\big) \overset{Law}{=} \big(X_{t_1}, X_{t_2}, ..., X_{t_n}\big).$$

From the invariance by time inversion property of Brownian motion, we get that for any $\tilde{t}_1, ..., \tilde{t}_n \geq 0$

$$\big(\tilde{t}_1 B_{1/\tilde{t}_1}, \tilde{t}_2 B_{1/\tilde{t}_2}, ..., \tilde{t}_n B_{1/\tilde{t}_n}\big) \overset{Law}{=} \big(B_{\tilde{t}_1}, B_{\tilde{t}_2}, ..., B_{\tilde{t}_n}\big).$$

Therefore, for $\tilde{t}_i := e^{-2t_i}$, $i := 1, ..., n$, we get that

$$\begin{aligned} \big(X_{-t_1}, X_{-t_2}, ..., X_{-t_n}\big) &= \big(e^{t_1} B_{e^{-2t_1}}, e^{t_2} B_{e^{-2t_2}}, ..., e^{t_n} B_{e^{-2t_n}}\big) \\ &\overset{Law}{=} \big(e^{-t_1} B_{e^{2t_1}}, e^{-t_2} B_{e^{2t_2}}, ..., e^{-t_n} B_{e^{2t_n}}\big) \\ &= \big(X_{t_1}, X_{t_2}, ..., X_{t_n}\big). \end{aligned}$$

Solution 2-2

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables with $X_n \sim \mathcal{N}(\mu_n, \sigma_n^2)$ for each $n \in \mathbb{N}$.

a) Since $(X_n)_{n\in\mathbb{N}}$ converges in distribution to X, we know from the continuity theorem for characteristic functions that for any $t \in \mathbb{R}$

$$\varphi_{X_n}(t) = \exp\Big(it\mu_n - \frac{t^2\sigma_n^2}{2}\Big) \longrightarrow \varphi_X(t) \quad \text{as } n \to \infty. \tag{1}$$

By taking absolute values, we see that

$$|\varphi_X(t)| = \lim_{n\to\infty} \exp\Big(-\frac{t^2\sigma_n^2}{2}\Big). \tag{2}$$

Moreover, φ_X is continuous in 0. Therefore, as $\varphi_X(0) = 1$, we can find $t_0 \neq 0$ such that $\varphi(t_0) \neq 0$. Taking the logarithm in (2), we see that the $\lim_{n\to\infty} \sigma_n^2$ exists and

$$\lim_{n\to\infty} \sigma_n^2 = -\frac{2}{t_0^2} \log|\varphi_X(t_0)| =: \sigma^2$$

As a consequence, due to (1), we see that the sequence

$$\exp\left(it\mu_n\right) = \exp\left(\frac{t^2\sigma_n^2}{2}\right)\varphi_{X_n}(t) \tag{3}$$

converges pointwise for any $t \in \mathbb{R}$ as n goes to infinity.
Next, we prove that the sequence $(\mu_n)_{n\in\mathbb{N}}$ converges. Set

$$\underline{\mu} := \liminf_{n\to\infty} \mu_n \quad \text{and} \quad \overline{\mu} := \limsup_{n\to\infty} \mu_n.$$

We claim that $\overline{\mu} < \infty$. Assume by contradiction that $\overline{\mu} = \infty$. In that case, we find a subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$ which diverge to infinity. For any point $a \in \mathbb{R}$ such that $P[X = a] = 0$, we deduce from the Portemonteau theorem of weak convergence that

$$\lim_{k\to\infty} P[X_{n_k} \leq a] = P[X \leq a].$$

Let $Y \sim \mathcal{N}(0,1)$. By definition of X_{n_k},

$$P[X_{n_k} \leq a] = P[\mu_{n_k} + \sigma_{n_k} Y \leq a].$$

By the divergence property of the sequence $(\mu_{n_k})_{k\in\mathbb{N}}$, since $(\sigma_{n_k})_{k\in\mathbb{N}}$ converges, we conclude that $\mu_{n_k} + \sigma_{n_k} Y$ converges P-a.s. to infinity. Thus, we get that $P[X \leq a] = 0$. But since we can find arbitrarily big points a satisfying $P[X = a] = 0$, we get a contradiction to the fact that $\lim_{a\to\infty} P[X \leq a] = 1$ by the definition of a cumulative distribution function. Thus, we conclude that $\overline{\mu} < \infty$. With a similar argument, one can show that $\underline{\mu} > -\infty$. Therefore, we deduce from the pointwise convergence of the sequence in (3) that for any $t \in \mathbb{R}$

$$\exp\left(it\underline{\mu}\right) = \exp\left(it\overline{\mu}\right).$$

Thus, we get that for any $t \in \mathbb{R}$

$$t(\overline{\mu} - \underline{\mu}) \equiv 0 \pmod{2\pi}$$

which implies that $\underline{\mu} = \overline{\mu}$. In other words, $\mu := \lim_{n\to\infty} \mu_n$ exists. As a consequence of (1), we get that for any $t \in \mathbb{R}$

$$\varphi_X(t) = \exp\left(it\mu - \frac{t^2\sigma^2}{2}\right)$$

and thus, $X \sim \mathcal{N}(\mu, \sigma^2)$.

b) Since $(X_n)_{n\in\mathbb{N}}$ converges in probability to X, $(X_n - X)_{n\in\mathbb{N}}$ converges in probability to 0 and hence $(X_n - X)_{n\in\mathbb{N}}$ converges in distribution to 0.
Fix any $n \in \mathbb{N}$. The sequence $(X_n - X_k)_{k\in\mathbb{N}}$ converges in probability to $X_n - X$ and hence $(X_n - X_k)_{k\in\mathbb{N}}$ converges in distribution to $X_n - X$. Now, since by assumption $(X_n)_{n\in\mathbb{N}}$ is a Gaussian process, we get that for each k, $X_n - X_k$ is normal distributed. Thus, we deduce from part a) that $X_n - X$ is normal distributed. Since $n \in \mathbb{N}$ was arbitrarily chosen, we get that $(X_n - X)_{n\in\mathbb{N}}$ is a sequence of Gaussian random variables. Moreover, since $(X_n - X)_{n\in\mathbb{N}}$ converges in distribution to 0, we deduce again from a) that

$$E[X_n - X] \longrightarrow 0 \quad \text{and} \quad \operatorname{Var}(X_n - X) \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

As a consequence, we get directly the L^2 convergence of X_n to X, since

$$\|X_n - X\|_{L^2}^2 = E\left[|X_n - X|^2\right] = \left(E[X_n - X]\right)^2 + \operatorname{Var}\left(X_n - X\right).$$

Solution 2-3

a) We know from the invariance by time inversion property of Brownian motion that the process $B = (B_t)_{t\geq 0}$ given by $B_0 = 0$ and $B_t = tX_{1/t}$ for $t > 0$ is a Brownian motion under W_0. Denoting the filtration generated by B by $(\mathcal{F}_t^B)_{t\geq 0}$, we see that for each $t > 0$

$$\mathcal{F}_{1/t}^B = \sigma\big(B_u, u \leq 1/t\big) = \sigma\big(X_u, u \geq t\big) = \widehat{\mathcal{F}}_t.$$

Therefore, we conclude that

$$\mathcal{F}_{0+}^B = \bigcap_{t>0} \mathcal{F}_{1/t}^B = \bigcap_{t>0} \widehat{\mathcal{F}}_t = \widehat{\mathcal{F}}.$$

Thus, by applying Blumenthal's 0-1 law to the (W_0-) Brownian motion B, we can conclude that $W_0[A] \in \{0, 1\}$ for every $A \in \mathcal{F}_{0+}^B = \hat{\mathcal{F}}$.

b) Let $A \in \widehat{\mathcal{F}}_1$. Then there is a set $A' \in \mathcal{F}$ such that $1_A = 1_{A'} \circ \theta_1$. For any $x \in \mathbb{R}$,

$$\begin{aligned} W_x[A] &= E_x[1_{A'} \circ \theta_1] = E_x[E_x[1_{A'} \circ \theta_1 | \mathcal{F}_1]] = E_x[E_{X_1}[1_{A'}]] \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} W_y[A'] \exp\left(-\frac{1}{2}(y-x)^2\right) dy \end{aligned} \tag{4}$$

where we use the Markov property of X in the third equality.
Now, let $A \in \widehat{\mathcal{F}}$. From a), we know that $W_0[A] \in \{0, 1\}$. Moreover, by definition, $A \in \widehat{\mathcal{F}}_1$. First, assume that $W_0[A] = 0$. Using (4) for $x = 0$, we obtain $W_y[A'] = 0$ for Lebesgue-a.e. y. Using (4) again shows that $W_x[A] = 0$ for all x. Second, if $W_0[A] = 1$ then $W_0[A^c] = 0$ and the above argument yields $W_x[A^c] = 0$ for all x. Thus, $W_x[A] = 1$ for all x.