ETH Zürich FS 2016
D-MATH
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Brownian Motion and Stochastic Calculus
Solution 3

Solution 3-1

Fix any $t, h \geq 0$ and $f \in b\mathcal{B}(\mathbb{R})$. The case where $h = 0$ is trivial, therefore, let $h > 0$. We know that Brownian motion is a Markov process with transition semigroup given by $R_0\tilde{f}(x) = \tilde{f}(x)$ and

$$R_h\tilde{f}(x) = \frac{1}{\sqrt{2\pi h}} \int_{\mathbb{R}} \tilde{f}(y) \exp\left(-\frac{(y-x)^2}{2h}\right) dy \quad \text{when} \quad h > 0, \quad \tilde{f} \in b\mathcal{B}(\mathbb{R}).$$

Therefore, we get for $\tilde{f}(x) := f(|x|) \in b\mathcal{B}(\mathbb{R})$ that

$$\begin{aligned}
&E\big[f(X_{t+h})\big|\mathcal{G}_t\big] \\
&= R_h\tilde{f}(B_t) \\
&= \frac{1}{\sqrt{2\pi h}} \int_{\mathbb{R}} \tilde{f}(y) \exp\left(-\frac{(y-B_t)^2}{2h}\right) dy \\
&= \frac{1}{\sqrt{2\pi h}} \int_{[0,\infty)} f(y) \exp\left(-\frac{(y-B_t)^2}{2h}\right) dy + \frac{1}{\sqrt{2\pi h}} \int_{(-\infty,0)} f(-y) \exp\left(-\frac{(y-B_t)^2}{2h}\right) dy \qquad (1)
\end{aligned}$$

By a change of variables and by observing that $\{0\}$ is a null set, we deduce from (1) that

$$\begin{aligned}
&E\big[f(X_{t+h})\big|\mathcal{G}_t\big] \\
&= \frac{1}{\sqrt{2\pi h}} \int_{[0,\infty)} f(y) \exp\left(-\frac{(y-B_t)^2}{2h}\right) dy + \frac{1}{\sqrt{2\pi h}} \int_{[0,\infty)} f(y) \exp\left(-\frac{(y+B_t)^2}{2h}\right) dy \qquad (2) \\
&= \tilde{R}_h f(B_t)
\end{aligned}$$

By symmetry of the expression in (2), we see that $E\big[f(X_{t+h})\big|\mathcal{G}_t\big] = \tilde{R}_h f(-B_t)$ and thus

$$E\big[f(X_{t+h})\big|\mathcal{G}_t\big] = \tilde{R}_h f(X_t).$$

Solution 3-2

a) Consider the set

$$A := \bigcap_{\varepsilon>0} \big\{ \sup_{0\leq s\leq\varepsilon} B_s > 0 \big\}$$

Clearly, $A \in \mathcal{F}_0^+$. Therefore, we deduce from Blumenthal's 0-1 law that $P[A] \in \{0,1\}$. By the inclusion $\{B_\varepsilon > 0\} \subseteq \{\sup_{0\leq s\leq\varepsilon} B_s > 0\}$ and as $B_\varepsilon \sim \mathcal{N}(0,\varepsilon)$, we conclude that

$$P\big[A\big] = \lim_{\varepsilon\downarrow 0} P\Big[\sup_{0\leq s\leq\varepsilon} B_s > 0\Big] \geq \lim_{\varepsilon\downarrow 0} P\big[B_\varepsilon > 0\big] \geq \frac{1}{2},$$

which implies that $P[A] = 1$.
Using this result to the Brownian motion $-B$, we obtain that

$$P\Big[\bigcap_{\varepsilon>0}\{\inf_{0\le s\le\varepsilon} B_s < 0\}\Big] = 1.$$

Thus, we showed that P-a.s.

$$\forall\varepsilon > 0, \quad \inf_{0\le s\le\varepsilon} B_s < 0 < \sup_{0\le s\le\varepsilon} B_s.$$

Therefore, for P-almost all $\omega \in \Omega$ we find a sequence $(s_n(\omega))_{n\in\mathbb{N}}$ in $(0, \infty)$ which is strictly decreasing and converges to 0 such that for any $k \in \mathbb{N}$

$$B_{s_{2k-1}(\omega)}(\omega) > 0 \quad \text{and} \quad B_{s_{2k}(\omega)}(\omega) < 0,$$

so the result is proved.

b) We observe that $Z(\omega)$ is the pre-image of the closed set $\{0\}$ of the map $t \mapsto B_t(\omega)$ which is for P-almost all $\omega \in \Omega$ continuous. Thus, for P-almost all $\omega \in \Omega$, $Z(\omega)$ is closed.
Since the map $t \mapsto B_t(\omega)$ is for P-almost all $\omega \in \Omega$ continuous, we deduce from a), by using the Intermediate Value Theorem, that 0 is an accumulation point in $Z(\omega)$.
Denote by $\lambda(\cdot)$ the Lebesgue measure on $\mathbb{R}$. Using Fubini and the fact that $P[B_t = 0] = 0$ for any $t \ge 0$, we see that

$$E\big[\lambda(Z)\big] = E\Big[\int_0^\infty 1_{\{B_t=0\}}\,dt\Big] = \int_0^\infty P[B_t = 0]\,dt = 0$$

and thus, $\lambda(Z(\omega)) = 0$ P-a.s.

Solution 3-3

a) Let $A \in \mathcal{F}_S$ and $t \ge 0$. Since $S \le T$ we see that $\{T \le t\} \subseteq \{S \le t\}$. Therefore, using the definition of $\mathcal{F}_S$ and of a stopping time, we see that

$$A \cap \{T \le t\} = \Big(A \cap \{S \le t\}\Big) \cap \{T \le t\} \in \mathcal{F}_t.$$

and thus $A \in \mathcal{F}_T$.
Now, $S \wedge T$ is a stopping time since $\{S \wedge T \le t\} = \{S \le t\} \cup \{T \le t\}$. Since $S \wedge T \le S$ and $S \wedge T \le T$, we deduce from above that $\mathcal{F}_{S\wedge T} \subseteq \mathcal{F}_S \cap \mathcal{F}_T$. For the converse, let $A \in \mathcal{F}_S \cap \mathcal{F}_T$. We see that

$$A \cap \{S \wedge T \le t\} = A \cap \Big(\{S \le t\} \cup \{T \le t\}\Big) = \Big(A \cap \{S \le t\}\Big) \cup \Big(A \cap \{T \le t\}\Big) \in \mathcal{F}_t$$

and thus, $A \in \mathcal{F}_{S\wedge T}$.

b) Let $t \ge 0$. By definition of stopping times, we observe that

$$\{S < T\} \cap \{T \le t\} = \bigcup_{q\in\mathbb{Q}\cap[0,t)} \Big(\{S \le q\} \cap \{T > q\} \cap \{T \le t\}\Big) \in \mathcal{F}_t.$$

Moreover, we see that

$$\begin{aligned}\{S<T\}\cap\{S\le t\} &= \bigcup_{q\in\mathbb{Q}_+}\Big(\{S\le q\}\cap\{T>q\}\cap\{S\le t\}\Big)\\ &= \Bigg(\bigcup_{q\in\mathbb{Q}\cap[0,t]}\{S\le q\}\cap\{T>q\}\Bigg)\cup\Bigg(\bigcup_{q\in\mathbb{Q}\cap(t,\infty)}\{S\le t\}\cap\{T>q\}\Bigg)\\ &= \Bigg(\bigcup_{q\in\mathbb{Q}\cap[0,t]}\{S\le q\}\cap\{T>q\}\Bigg)\cup\Bigg(\{S\le t\}\cap\{T>t\}\Bigg)\in\mathcal{F}_t.\end{aligned}$$

Thus, we conclude that $\{S<T\}\in\mathcal{F}_S\cap\mathcal{F}_T$. Moreover, by symmetry, we get that

$$\{S\le T\}=\{T<S\}^c\in\mathcal{F}_S\cap\mathcal{F}_T.$$

Now, let $A\in\mathcal{F}_S$. From above, we know that $\{S\le T\}\in\mathcal{F}_S\cap\mathcal{F}_T$. Thus $A\cap\{S\le T\}\in\mathcal{F}_S$. Moreover, we see that

$$\Big(A\cap\{S\le T\}\Big)\cap\{T\le t\}=\Big(A\cap\{S\le t\}\Big)\cap\Big(\{S\le T\}\cap\{T\le t\}\Big)\in\mathcal{F}_t$$

and so $A\cap\{S\le T\}\in\mathcal{F}_T$. Thus, we deduce from above and a) that

$$A\cap\{S\le T\}\in\mathcal{F}_S\cap\mathcal{F}_T=\mathcal{F}_{S\wedge T}.$$

With the same argument, we see that for $A\in\mathcal{F}_S$, we have $A\cap\{S<T\}\in\mathcal{F}_{S\wedge T}$.