ETH Zürich FS 2016
D-MATH
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Brownian Motion and Stochastic Calculus
Solution 4

Solution 4-1

For any $g : \mathbb{R} \to \mathbb{R}$ bounded Borel measurable function, we know that

$$E\big[g(|B_t|)\big] = \int_{-\infty}^{\infty} g(|x|) \frac{1}{\sqrt{2\pi t}}\, e^{-x^2/(2t)}\, dx = \int_0^{\infty} g(x)\, \sqrt{\frac{2}{\pi t}}\, e^{-x^2/(2t)}\, dx \tag{1}$$

Thus, we see that the probability density function of $|B_t|$ on $\mathbb{R}$ is given by the function

$$x \mapsto \mathbf{1}_{x\geq 0} \sqrt{\frac{2}{\pi t}}\, e^{-x^2/(2t)}.$$

From (2.55) of the script, we know that the probability density function of the joint law of (B_t, M_t) where $M_t := \sup_{0\leq s\leq t} B_s$ is given by the function

$$(x, y) \mapsto \frac{2(2y - x)}{\sqrt{2\pi t^3}}\, e^{-(2y-x)^2/(2t)}\, \mathbf{1}_{\{y\geq 0,\, x\leq y\}}. \tag{2}$$

Take any $g : \mathbb{R} \to \mathbb{R}$ bounded Borel measurable function. We deduce from (2) that

$$E\big[g(M_t - B_t)\big] = \int\int_{0\leq y,\, 0\leq y-x} g(y - x)\, \frac{2(2y - x)}{\sqrt{2\pi t^3}}\, e^{-(2y-x)^2/(2t)}\, dx\, dy.$$

By a change of variable $u := y - x$ $v := y$ we get that

$$E\big[g(M_t - B_t)\big] = \int\int_{0\leq u,\, 0\leq v} g(u)\, \sqrt{\frac{2}{\pi t^3}}\, (u + v)\, e^{-(u+v)^2/(2t)}\, du\, dv \tag{3}$$

By another change of variable $n := u$ and $m := u + v$ and as $\int x\, e^{-cx^2/2}\, dx = -\frac{e^{-cx^2/2}}{c}$, we get that

$$\begin{aligned} E\big[g(M_t - B_t)\big] &= \int_0^{\infty} g(n)\, \sqrt{\frac{2}{\pi t^3}} \int_n^{\infty} m\, e^{-m^2/(2t)}\, dm\ dn \\ &= \int_0^{\infty} g(n)\, \sqrt{\frac{2}{\pi t}}\, e^{-n^2/(2t)}\, dn. \end{aligned} \tag{4}$$

Comparing (1) with (4) yields that $M_t - B_t \overset{Law}{=} |B_t|$.
The fact that $M_t \overset{Law}{=} |B_t|$ is a consequence of (2.48) in the lecture notes.

Solution 4-2

We observe that by the simple Markov property

$$W_0\Big(S \le s\Big) = W_0\Big(H_0 \circ \theta_s > 1-s\Big) = E_0\Big[W_{X_s}\Big(H_0 > 1-s\Big)\Big] \overset{\text{symmetry}}{=} E_0\Big[W_0\Big(H_{X_s(\tilde\omega)} > 1-s\Big)\Big].$$

Therefore, Using the law of H_{X_s} under W_0 (see (2.54) in the lecture notes), Fubini and that $\int x\, e^{-cx^2/2}\, dx = -\frac{e^{-cx^2/2}}{c}$, we get that

$$\begin{aligned}
W_0\Big(S \le s\Big) &= E_0\Big[W_0\Big(H_{X_s(\tilde\omega)} > 1-s\Big)\Big] \\
&\overset{(2.54)}{=} E_0\Big[\int_{1-s}^{\infty} \frac{|X_s|}{\sqrt{2\pi l^3}}\, e^{-|X_s|^2/(2l)}\, dl\Big] \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi s}}\, e^{-a^2/(2s)} \int_{1-s}^{\infty} \frac{|a|}{\sqrt{2\pi l^3}}\, e^{-a^2/(2l)}\, dl\ \ da \\
&= 2\int_{0}^{\infty} \frac{1}{\sqrt{2\pi s}}\, e^{-a^2/(2s)} \int_{1-s}^{\infty} \frac{a}{\sqrt{2\pi l^3}}\, e^{-a^2/(2l)}\, dl\ \ da \\
&= 2\int_{1-s}^{\infty} \frac{1}{2\pi\,\sqrt{sl^3}}\, \Big(\int_0^{\infty} a\, e^{-\frac{a^2}{2}(\frac{1}{s}+\frac{1}{l})}\, da\Big)\, dl \\
&= 2\int_{1-s}^{\infty} \frac{1}{2\pi\,\sqrt{sl^3}}\, \Big(\frac{1}{s}+\frac{1}{l}\Big)^{-1}\, dl \\
&= \frac{1}{\pi}\int_{1-s}^{\infty} \sqrt{\frac{s}{l}}\, \frac{1}{s+l}\, dl \\
&\overset{w=\frac{l}{s}}{=} \frac{1}{\pi}\int_{\frac{1}{s}-1}^{\infty} \frac{1}{\sqrt{w}}\, \frac{1}{1+w}\, dw \\
&\overset{v=\frac{1}{1+w}}{=} \frac{1}{\pi}\int_0^{s} \frac{dv}{\sqrt{v(1-v)}} \\
&= \frac{2}{\pi}\arcsin\sqrt{s}.
\end{aligned}$$

Solution 4-3

a) Take any $\alpha > \frac{1}{2}$ and let $M \in \mathbb{N}$ satisfying $M(\alpha - \frac{1}{2}) > 1$. If $B_.(\omega)$ is Hölder-continuous of order α at the point $s \in [0,1]$, there exists a constant C so that $|B_t(\omega) - B_s(\omega)| \le C|t-s|^\alpha$ for t near s. Then $|B_{\frac{k}{n}}(\omega) - B_{\frac{k-1}{n}}(\omega)| \le Cn^{-\alpha}$ for all large enough n, for $\frac{k}{n}$ near s and M successive k's. The set $\{B_.(\omega)$ is α-Hölder at some $s \in [0,1]\}$ is therefore contained in

$$\bigcup_{C\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \bigcap_{n\ge m} \bigcup_{k=0,\dots,n-1} \bigcap_{j=1}^{M} \Big\{ |B_{\frac{k+j}{n}}(\omega) - B_{\frac{k+j-1}{n}}(\omega)| \le C\frac{1}{n^\alpha} \Big\}; \qquad (*)$$

We show that this is a nullset. As the above Brownian increments are iid $\sim N(0,\frac{1}{n})$, we have, with $Z \sim N(0,1)$, as $P[|Z| \le \varepsilon] \le \varepsilon$ for any $\varepsilon \ge 0$ (see (1)), that

$$P\Big[\bigcap_{i=1}^{M} \Big\{ |B_{\frac{k+j}{n}}(\omega) - B_{\frac{k+j-1}{n}}(\omega)| \le C\frac{1}{n^\alpha} \Big\}\Big] = \Big(P\Big[|Z| \le \frac{C}{n^{\alpha-1/2}}\Big]\Big)^M \le C^M n^{-M(\alpha-\frac{1}{2})}. \quad (5)$$

Now, we have

$$
\begin{aligned}
D_m :&= \bigcap_{n\geq m} \bigcup_{k=0,\dots,n-1} \bigcap_{j=1}^{M} \left\{ |B_{\frac{k+j}{n}}(\omega) - B_{\frac{k+j-1}{n}}(\omega)| \leq C\frac{1}{n^\alpha} \right\} \\
&\subseteq \bigcup_{k=0,\dots,n-1} \bigcap_{j=1}^{M} \left\{ |B_{\frac{k+j}{n}}(\omega) - B_{\frac{k+j-1}{n}}(\omega)| \leq C\frac{1}{n^\alpha} \right\} \quad \text{for each } n \geq m
\end{aligned}
$$

and therefore, due to (5), as $M(\alpha - \frac{1}{2}) > 1$, we get

$$
\begin{aligned}
P\big[D_m\big] &\leq \limsup_{n\to\infty} P\left[\bigcup_{k=0,\dots,n-1} \bigcap_{j=1}^{M} \left\{ |B_{\frac{k+j}{n}}(\omega) - B_{\frac{k+j-1}{n}}(\omega)| \leq C\frac{1}{n^\alpha} \right\} \right] \\
&\leq \limsup_{n\to\infty} n\, C^M\, n^{-M(\alpha-\frac{1}{2})} \\
&= 0.
\end{aligned}
$$

Therefore, being a countable union of nullsets, $P[(*)] = 0$.

b) Let $Y_\sigma \sim \mathcal{N}(0, \sigma^2)$ for any $\sigma \geq 0$. We note that $E[Y_\sigma^m] = C\sigma^m$, where $C = E[Y_1^m]$. Thus

$$
E\big[|B_t - B_s|^{2n}\big] = C_n |t-s|^n \quad \text{for all } n.
$$

Writing $\alpha_n := 2n$ and $\beta_n := n-1$ yields that $\frac{\beta_n}{\alpha_n} < \frac{1}{2}$ for any $n \in \mathbb{N}$. Moreover, $\frac{\beta_n}{\alpha_n}$ converges to $1/2$. Thus, we get the result applying the *Kolmogorov-Čentsov theorem*.

Remark: In fact, $E[Y_\sigma^n] = (n-1)!!\,\sigma^n$ for n even and 0 otherwise, where $n!!$ denotes the double factorial, that is the product of every odd number from n to 1.