Brownian Motion and Stochastic Calculus
Solution 5

Solution 5-1

a) Since $X_t \sim \mathcal{N}(x,t)$ under W_x,

$$W_x(X_t < 0) = \Phi(-\frac{x}{\sqrt{t}}).$$

Thus, by continuity of the normal distribution,

$$\lim_{t\to\infty} W_x(X_t < 0) = \frac{1}{2}.$$

Therefore, since Brownian motion paths are a.s. continuous, we get that for all $x > 0$,

$$W_x(H_0 < \infty) \geq \frac{1}{2}.$$

Applying symmetry, we can extend the result to the case $x < 0$. Finally, $W_0(H_0 < \infty) = 1$ by definition of H_0 and Brownian motion.
Thus, directly from the simple Markov property, we get

$$W_x\big(X_t = 0 \text{ for some } t \geq n\big) = E_x\big[W_{X_n}(H_0 < \infty)\big] \geq \frac{1}{2}.$$

Taking the limit $n \to \infty$, it follows that $W_x(A) \geq \frac{1}{2}$ for any x.

b) We know from part a) that

$$\lim_{n\to\infty} W_x(X_t = 0 \text{ for some } t \geq n) = W_x(A) \geq \frac{1}{2}.$$

Now, observe that A is an element of the asymptotic σ-field, i.e. $A \in \widehat{\mathcal{F}} := \bigcap_{t\geq 0} \widehat{\mathcal{F}}_t$, where, $\widehat{\mathcal{F}}_t := \sigma(X_u,\ u \geq t)$ for $t \geq 0$. Therefore, we conclude from Exercise 2.3 that either $W_x(A) = 0$ for each $x \in \mathbb{R}$ or $W_x(A) = 1$ for each $x \in \mathbb{R}$ and so $W_x(A) = 1$ for all x.

Solution 5-2

<u>*a)* $\Longrightarrow$ *b)* :</u> Fix any $\lambda \in \mathbb{R}$. For any $t \geq 0$ we see directly that M_t^λ is $\mathcal{F}_t$-measurable as B_t is, and is integrable as it is dominated by the constant $e^{\lambda^2 t/2}$. Now, for any $0 \leq s \leq t < \infty$, we can write

$$E[M_t^\lambda \,|\, \mathcal{F}_s] = E\left[M_s^\lambda \exp\left(i\,\lambda\,(B_t - B_s) + \frac{\lambda^2\,(t-s)}{2}\right)\middle|\mathcal{F}_s\right].$$

Note that M_s^λ is measurable w.r.t. $\mathcal{F}_s$ and $B_t - B_s$ is independent of $\mathcal{F}_s$, hence

$$E[M_t^\lambda \,|\, \mathcal{F}_s] = M_s^\lambda \exp\left(\frac{\lambda^2\,(t-s)}{2}\right) E\Big[\exp\big(i\,\lambda\,(B_t - B_s)\big)\Big] \tag{1}$$

Since $B_t - B_s$ has law $\mathcal{N}(0, t-s)$, we have

$$E\Big[\exp\big(i\,\lambda\,(B_t - B_s)\big)\Big] = \exp\left(-\frac{\lambda^2\,(t-s)}{2}\right) \tag{2}$$

Putting (2) into (1) yields $E[M_t^\lambda \,|\, \mathcal{F}_s] = M_s^\lambda$.

$\underline{b) \Longrightarrow a)}$: By assumption, (M_t^λ) is a martingale, hence for all $\lambda \in \mathbb{R}, 0 \leq s \leq t$, we have $E[M_t^\lambda \,|\, \mathcal{F}_s] = M_s^\lambda$. Moreover,

$$E[M_t^\lambda \,|\, \mathcal{F}_s] = E\left[M_s^\lambda \exp\left(i\,\lambda\,(B_t - B_s) + \frac{\lambda^2\,(t-s)}{2}\right) \middle|\, \mathcal{F}_s\right].$$

Since (B_t) is $\mathcal{F}_t$ adapted, M_s^λ is measurable w.r.t. $\mathcal{F}_s$, hence

$$E[M_t^\lambda \,|\, \mathcal{F}_s] = M_s^\lambda \exp\left(\frac{\lambda^2\,(t-s)}{2}\right) E\left[\exp\Big(i\,\lambda\,(B_t - B_s)\Big) \middle|\, \mathcal{F}_s\right].$$

Since $E[M_t^\lambda \,|\, \mathcal{F}_s] = M_s^\lambda$, we have

$$E\left[\exp\Big(i\,\lambda\,(B_t - B_s)\Big) \middle|\, \mathcal{F}_s\right] = \exp\left(-\frac{\lambda^2\,(t-s)}{2}\right). \tag{3}$$

Taking expectation again at both sides yields

$$E\left[\exp\Big(i\,\lambda\,(B_t - B_s)\Big)\right] = \exp\left(-\frac{\lambda^2\,(t-s)}{2}\right). \tag{4}$$

Thus, by the property of the characteristic functions, we see directly that the random variable $B_t - B_s$ has law $\mathcal{N}(0, t-s)$. Moreover, we see that the random variable $B_t - B_s$ is independent of $\mathcal{F}_s$. Indeed, for any random variable X which is $\mathcal{F}_s$-measurable, let us compute the joint characteristic function of $(B_t - B_s, X)$. For all $\lambda, \mu \in \mathbb{R}$,

$$\begin{aligned} E\Big[\exp\big(i\,\lambda\,(B_t - B_s)\big)\,\exp(i\mu X)\Big] &= E\Big[E\Big[\exp\big(i\,\lambda\,(B_t - B_s)\big) \,\Big|\, \mathcal{F}_s\Big]\,\big[\exp(i\mu X)\big]\\ &\overset{\text{by (3)}}{=} \exp\left(-\frac{\lambda^2\,(t-s)}{2}\right) E\big[\exp(i\mu X)\big]\\ &\overset{\text{by (4)}}{=} E\Big[\exp\big(i\,\lambda\,(B_t - B_s)\big)\Big]\,E\big[\exp(i\mu X)\big]. \end{aligned}$$

So we get the conclusion.

Solution 5-3

a) For any $n \geq m$, $N, M > 0$, by the backward submartingale property, using that the set $\{-X_n > M\} \in \mathcal{F}_n$, we obtain that

$$\begin{aligned}
E\big[-X_n \mathbf{1}_{\{-X_n \geq M\}}\big] &= E\big[-X_n\big] - E[-X_n \mathbf{1}_{\{-X_n < M\}}\big] \\
&\leq E\big[-X_n\big] - E[-X_m \mathbf{1}_{\{-X_n < M\}}\big] \\
&= E\big[-X_n\big] - E\big[-X_m\big] + E[-X_m \mathbf{1}_{\{-X_n \geq M\}}\big] \\
&= E\big[X_m - X_n\big] + E[-X_m \mathbf{1}_{\{-X_n \geq M, -X_m \geq N\}}\big] \\
&\quad + E[-X_m \mathbf{1}_{\{-X_n \geq M, 0 < -X_m < N\}}\big] + E[-X_m \mathbf{1}_{\{-X_n \geq M, -X_m \leq 0\}}\big] \\
&\leq E\big[X_m - X_n\big] + E[-X_m \mathbf{1}_{\{-X_n \geq M, -X_m \geq N\}}\big] + E[-X_m \mathbf{1}_{\{-X_n \geq M, 0 < -X_m < N\}}\big] \\
&\leq E\big[X_m\big] - E\big[X_n\big] + E\big[|X_m| \mathbf{1}_{\{-X_m \geq N\}}\big] + \frac{N}{M} E\big[-X_n \mathbf{1}_{\{-X_n \geq M\}}\big] \\
&\leq E\big[X_m\big] - E\big[X_n\big] + E\big[|X_m| \mathbf{1}_{\{-X_m \geq N\}}\big] + \frac{N}{M} E\big[-X_n \mathbf{1}_{\{-X_n \geq 0\}}\big] \\
&= E\big[X_m\big] - E\big[X_n\big] + E\big[|X_m| \mathbf{1}_{\{-X_m \geq N\}}\big] + \frac{N}{M} E\big[X_n^-\big].
\end{aligned}$$

b) (i) First, as $(X_n)_{n\in\mathbb{N}}$ is a backward submartingale, we obtain that for any n

$$X_n^+ = X_n + X_n^- \leq E[X_0 \,|\mathcal{F}_n] + X_n^-$$

Thus, as $(E[X_0 \,|\mathcal{F}_n])_{n\in\mathbb{N}}$ is uniformly integrable, if we can show that $(X_n^-)_{n\in\mathbb{N}}$ is uniformly integrable, we can conclude that also $(X_n^+)_{n\in\mathbb{N}}$ is uniformly integrable which then implies that $(X_n)_{n\in\mathbb{N}}$ is uniformly integrable.

(ii) Now, we prove that $\sup_{n\in\mathbb{N}} E[X_n^-] < \infty$. We first observe that $(X_n^+)_{n\in\mathbb{N}}$ is also a backward submartingale. Indeed, due to Jensen's inequality, we obtain that

$$E\big[X_n^+ \,\big|\, \mathcal{F}_{n+1}\big] \geq E[X_n \,|\, \mathcal{F}_{n+1}]^+ \geq X_{n+1}^+.$$

Assume by contradiction that $\sup_{n\in\mathbb{N}} E[X_n^-] = \infty$. Then, we can find a subsequence $(n_k)_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} E[X_{n_k}^-] = \sup_{n\in\mathbb{N}} E[X_n^-] = \infty.$$

Knowing that $(X_n^+)_{n\in\mathbb{N}}$ is a backward submartingale, we obtain for any k that

$$E\big[X_{n_k}^-\big] = E\big[X_{n_k}^+\big] - E\big[X_{n_k}\big] \leq E\big[X_0^+\big] - E\big[X_{n_k}\big] \leq E\big[|X_0|\big] - E\big[X_{n_k}\big].$$

But as X_0 is integrable and as $\lim_{k\to\infty} E\big[X_{n_k}\big] > -\infty$ by assumption, we obtain that

$$\infty = \lim_{k\to\infty} E[X_{n_k}^-] \leq E\big[|X_0|\big] - \lim_{k\to\infty} E\big[X_{n_k}\big] < \infty$$

which gives us a contradiction. Thus we have proved that $\sup_{u\in\mathbb{N}} E[X_u^-] < \infty$.

(iii) $(X_n^-)_{n\in\mathbb{N}}$ is uniformly integrable if and only if for any $\varepsilon > 0$ we can find $M \geq 0$ such that

$$\sup_{n\in\mathbb{N}} E\big[-X_n \mathbf{1}_{\{-X_n \geq M\}}\big] \leq \varepsilon.$$

Now note that

$$\sup_{n\in\mathbb{N}} E\big[-X_n \mathbf{1}_{\{-X_n \geq M\}}\big] \leq \max_{n<m} E\big[-X_n \mathbf{1}_{\{-X_n \geq M\}}\big] \vee \sup_{n\geq m} E\big[-X_n \mathbf{1}_{\{-X_n \geq M\}}\big].$$

Since X_n is integrable for every $n \in \mathbb{N}$, for each m, we can find $M_m \geq 0$ such that for any $M \geq M_m$

$$\max_{0\leq n\leq m} E\big[-X_n \mathbf{1}_{\{-X_n \geq M\}}\big] \leq \varepsilon.$$

This is because we are looking at the maximum over finitely many terms.
Now it suffices to find $M \geq M_m$ such that

$$\sup_{n \geq m} E\big[-X_n \, \mathbf{1}_{\{-X_n \geq M\}}\big] \leq \varepsilon.$$

Fix any $\varepsilon > 0$. The assumption that $\lim_{n\to\infty} E[X_n] > -\infty$ implies that $E[X_n]$ is a Cauchy sequence. Hence we can find $m \in \mathbb{N}$ such that for any $n \geq m$

$$E[X_m] - E[X_n] \leq \varepsilon/3.$$

For any $n \geq m$, $N, M > 0$, by a), by the choice of m and by (b ii) as $\sup_{k\in\mathbb{N}} E[X_k^-] < \infty$, we obtain that

$$\begin{aligned} E\big[-X_n \, \mathbf{1}_{\{-X_n \geq M\}}\big] &\leq E\big[X_m\big] - E\big[X_n\big] + E\big[|X_m| \, \mathbf{1}_{\{-X_m \geq N\}}\big] + \frac{N}{M} E\big[X_n^-\big] \\ &\leq \varepsilon/3 + E\big[|X_m| \, \mathbf{1}_{\{-X_m \geq N\}}\big] + \frac{N}{M} E\big[X_n^-\big] \\ &\leq \varepsilon/3 + E\big[|X_m| \, \mathbf{1}_{\{-X_m \geq N\}}\big] + \frac{N}{M} \sup_{u\in\mathbb{N}} E\big[X_u^-\big]. \end{aligned}$$

Since X_u is integrable for any $u \in \mathbb{N}$ we can find N big enough such that second term above is smaller than $\varepsilon/3$. After having chosen N we can find $M \geq M_m$ such that that the last term above is smaller than $\varepsilon/3$. Thus, we get for that chosen M that

$$\sup_{n \geq m} E\big[-X_n \, \mathbf{1}_{\{-X_n \geq M\}}\big] \leq \varepsilon$$

hence we get the result.