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D-MATH
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Brownian Motion and Stochastic Calculus
Solution 6

Solution 6-1

Let us first recall the statement of the discrete **optional stopping theorem**:

If $L \le M$ are stopping times and $Y_{M\wedge n}$ is a uniformly integrable submartingale, then $EY_L \le EY_M$ and

$$Y_L \le E(Y_M|\mathcal{F}_L).$$

As a consequence, when L, M are bounded, we have $Y_L \le E(Y_M|\mathcal{F}_L)$.

Consider the sequence of stopping times $(S_n)_{n\in\mathbb{N}}$ defined by

$$S_n(\omega) := \sum_{k=1}^{\infty} \frac{k}{2^n}\, \mathbf{1}_{\{\frac{k-1}{2^n} \le S(\omega) < \frac{k}{2^n}\}}$$

and define the sequence of stopping times $(T_n)_{n\in\mathbb{N}}$ in a similar way. They are indeed $\mathcal{G}_t$-stopping times since

$$\{S_n \le t\} = \bigcup_{k:\ k2^{-n} \le t} \{(k-1)2^{-n} \le S < k2^{-n}\} \ \in \mathcal{G}_t.$$

Moreover, both sequences $(S_n)_{n\in\mathbb{N}}$ and $(T_n)_{n\in\mathbb{N}}$ decrease and converge to S and T respectively, when we let n go to infinity, due to the fact that they are bounded so $S, T \ne \infty$. Furthermore, for any fixed $n \in \mathbb{N}$, both S_n and T_n take on a countable number of values and we also have $S_n \le T_n$. Therefore, by the discrete stopping theorem, we get for any $A \in \mathcal{G}_{S_n}$ that

$$E\big[X_{S_n} 1_A\big] \le E\big[X_{T_n} 1_A\big] \tag{1}$$

As $\mathcal{G}_S \subseteq \mathcal{G}_{S_n}$ for any $n \in \mathbb{N}$, since $S \le S_n$, we conclude that (1) holds true for any $A \in \mathcal{G}_S$. Again, as for any n, S_n takes on a countable number of values, by the discrete stopping theorem, $(X_{S_n})_{n\in\mathbb{N}}$ is a backward submartingale with respect to $(\mathcal{G}_{S_n})_{n\in\mathbb{N}}$, where the sequence $(E[X_{S_n}])_{n\in\mathbb{N}}$ is decreasing and bounded from below by $E[X_0]$. The same holds true for $(X_{T_n})_{n\in\mathbb{N}}$. We conclude from Series 5 exercise 3 b) that both sequences of random variables $(X_{S_n})_{n\in\mathbb{N}}$ and $(X_{T_n})_{n\in\mathbb{N}}$ are uniformly integrable. Now, by right-continuity of X we get that $\lim_{n\to\infty} X_{S_n} = X_S$ as well as $\lim_{n\to\infty} X_{T_n} = X_T$. Therefore, due to the uniform integrability, we get that also $\lim_{n\to\infty} E[|X_{S_n}|] = E[|X_S|]$ as well as $\lim_{n\to\infty} E[|X_{T_n}|] = E[|X_T|]$. Moreover, using Fatou, we obtain for any $M > 0$ that

$$E\big[|X_T|\big] \le \liminf_{n\to\infty} E\big[|X_{T_n}|\big] \le M + \sup_{n\in\mathbb{N}} E\big[|X_{T_n}|\, \mathbf{1}_{\{|X_{T_n}| \ge M\}}\big]$$

which is finite for big enough M by the uniform integrability of $(X_{T_n})_{n\in\mathbb{N}}$. Similarly, we see that X_S is integrable, too. Moreover, for any $A \in \mathcal{G}_S$ due to (1) and the uniform integrability, we obtain that

$$E\big[X_S\, 1_A\big] = \lim_{n\to\infty} E\big[X_{S_n}\, 1_A\big] \le \lim_{n\to\infty} E\big[X_{T_n}\, 1_A\big] = E\big[X_T\, 1_A\big],$$

which proves that

$$E\left[X_T \,\middle|\, \mathcal{G}_S\right] \geq X_S \quad \text{a.s.}$$

Remark: With a similar argument, one can show that the stopping theorem also holds for general (i.e. non-bounded) stopping times, but one has to assume additionally that the submartingale is uniformly integrable. On the other hand, the assumptions of the right-continuity of the submartingale is crucial.

Solution 6-2

Without loss of generality, assume that $M_0 = 0$. Suppose first that M has variation on $[0,t]$, denoted by $\mathrm{Var}_t(M)$, which is uniformly bounded, i.e. assume that

$$\exists K \geq 0 \text{ such that } P\text{-a.s.}, \quad \forall t \geq 0, \quad \mathrm{Var}_t(M.(\omega)) \leq K. \tag{2}$$

Fix any $t \geq 0$. Consider a subdivision σ of the interval $[0,t]$ given by: $0 = t_0 < t_1 < ... < t_n = t$. We define its mesh size by:

$$\|\sigma\| := \max_{0 \leq i \leq n-1} |t_{i+1} - t_i|.$$

We claim that by the martingale property of M we have for any $0 \leq i \leq n-1$ that

$$E\Big[\big(M_{t_{i+1}} - M_{t_i}\big)^2\Big] = E\Big[M_{t_{i+1}}^2 - M_{t_i}^2\Big]. \tag{3}$$

Indeed, we get by applying the martingale property that

$$\begin{aligned} E\Big[\big(M_{t_{i+1}} - M_{t_i}\big)^2 \,\Big|\, \mathcal{F}_{t_i}\Big] &= E\Big[M_{t_{i+1}}^2 \,\Big|\, \mathcal{F}_{t_i}\Big] - 2M_{t_i}\, E\Big[M_{t_{i+1}} \,\Big|\, \mathcal{F}_{t_i}\Big] + M_{t_i}^2 \\ &= E\Big[M_{t_{i+1}}^2 \,\Big|\, \mathcal{F}_{t_i}\Big] - M_{t_i}^2. \end{aligned}$$

By taking the expectation in the above equality, we proved the claim. Therefore, we deduce from (3) that

$$E\Big[M_t^2\Big] = E\Big[\sum_{i=0}^{n-1} \big(M_{t_{i+1}} - M_{t_i}\big)^2\Big].$$

Thus, due to our assumption (2), we get

$$E\Big[M_t^2\Big] \leq E\Big[\mathrm{Var}_t(M) \max_{0 \leq i \leq n-1} |M_{t_{i+1}} - M_{t_i}|\Big] \leq KE\Big[\max_{0 \leq i \leq n-1} |M_{t_{i+1}} - M_{t_i}|\Big] \tag{4}$$

Now, take any sequence $(\sigma_k)_{k \in \mathbb{N}}$ of subdivisions of $[0,t]$ with $\lim_{k \to \infty} \|\sigma_k\| = 0$. Using (4), we deduce from the continuity of M (and so uniform continuity of M on $[0,t]$) and by using the dominated convergence theorem (which we can use by the assumption (2)) , that

$$E\big[M_t^2\big] = 0 \quad \text{which implies that} \quad M_t^2 = 0 \;\; P\text{-a.s.}$$

Since $t \geq 0$ was arbitrarily chosen, we obtain that

$$P\text{-a.s.}, \;\; \forall t \in \mathbb{Q}_+, \;\; M_t = 0.$$

Using the continuity of M we obtain that

$$P\text{-a.s.}, \;\; \forall t \geq 0, \;\; M_t = 0.$$

Now, let M be a continuous martingale of finite variation starting at 0 <u>without</u> satisfying the additional assumption (2). Consider for any $k \in \mathbb{N}$ the stopping time

$$\tau_k := \inf\big\{t \geq 0 \,\big|\, \mathrm{Var}_t(M) \geq k\big\}.$$

As M is an adapted continuous process, $\mathrm{Var}(M)$ is continuous and adapted, too. Hence it is easy to check that for any k, τ_k is a stopping time. Moreover, τ_k converges to infinity as k goes to infinity, as M is of finite variation. Moreover, for any k, the stopped process $M_t^{\tau_k} = (M_t^{\tau_k})_{t \geq 0}$ is a continuous martingale of finite variation starting at 0 which satisfies the additional condition (2) (for the constant $K = k$). Thus, from the above result, we obtain that for any $k \in \mathbb{N}$

$$P\text{-a.s.}, \;\; \forall t \geq 0, \;\; M_t^{\tau_k} = 0.$$

Thus, letting k goes to infinity, we obtain the desired result.

Solution 6-3

a) We know that for any $\lambda \in \mathbb{R}$, the process $M^\lambda := (M_t^\lambda)_{t\geq 0}$ defined by

$$M_t^\lambda = \exp\Big(\lambda B_t - \frac{\lambda^2}{2}t\Big)$$

is a continuous $(\mathcal{F}_t)$-martingale, where we denote by $(\mathcal{F}_t)_{t\geq 0}$ the filtration generated by B. Moreover, for any $n \in \mathbb{N}$, $T_a \wedge n$ is a bounded stopping time. Thus, applying the stopping theorem, we get

$$E\Big[M^\lambda_{T_a\wedge n}\,\Big|\,\mathcal{F}_0\Big] = M_0^\lambda = 1 \quad P\text{-a.s.}$$

By taking expectations, we get that

$$E\Big[M^\lambda_{T_a\wedge n}\Big] = 1$$

Now, on the event $\{T_a < \infty\}$ we have

$$\exp\Big(\lambda B_{T_a\wedge n} - \frac{\lambda^2}{2}(T_a \wedge n)\Big) \overset{n\to\infty}{\longrightarrow} \exp\Big(\lambda B_{T_a} - \frac{\lambda^2}{2}T_a\Big) = e^{\lambda a}\,\exp\Big(-\frac{\lambda^2}{2}T_a\Big).$$

On the event $\{T_a = \infty\}$ we have $B_t \leq a$ for any $t \geq 0$ and thus we get for any $\lambda > 0$ that

$$\exp\Big(\lambda B_{T_a\wedge n} - \frac{\lambda^2}{2}(T_a \wedge n)\Big) = \exp\Big(\lambda B_n - \frac{\lambda^2}{2}n\Big) \overset{n\to\infty}{\longrightarrow} 0.$$

We conclude that for any $\lambda > 0$

$$\exp\Big(\lambda B_{T_a\wedge n} - \frac{\lambda^2}{2}(T_a \wedge n)\Big) \overset{n\to\infty}{\longrightarrow} e^{\lambda a}\,\exp\Big(-\frac{\lambda^2}{2}T_a\Big)\,\mathbf{1}_{\{T_a<\infty\}} \quad P\text{-a.s.} \tag{5}$$

Observe that for any $n \in \mathbb{N}$ we have

$$0 \leq \exp\Big(\lambda B_{T_a\wedge n} - \frac{\lambda^2}{2}(T_a \wedge n)\Big) \leq e^{\lambda a}$$

Thus, we deduce from (5), by applying the dominated convergence theorem, that for any $\lambda > 0$

$$1 = E\big[M^\lambda_{T_a\wedge n}\big] \overset{n\to\infty}{\longrightarrow} e^{\lambda a}\,E\Big[\exp\Big(-\frac{\lambda^2}{2}T_a\Big)\,\mathbf{1}_{\{T_a<\infty\}}\Big]$$

and so, for any $\lambda > 0$

$$e^{\lambda a}\,E\Big[\exp\Big(-\frac{\lambda^2}{2}T_a\Big)\,\mathbf{1}_{\{T_a<\infty\}}\Big] = 1. \tag{6}$$

Take any positive, decreasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ converging to 0. We deduce from (6) and the monotone convergence theorem that

$$P\big[T_a < \infty\big] = \lim_{n\to\infty} E\Big[\exp\Big(-\frac{\lambda_n^2}{2}T_a\Big)\,\mathbf{1}_{\{T_a<\infty\}}\Big] = \lim_{n\to\infty} e^{-\lambda_n a} = 1$$

which proves the first part. Thus, as we now know that $P[T_a < \infty] = 1$ we get from (6) that for any $\lambda > 0$

$$e^{\lambda a}\,E\Big[\exp\Big(-\frac{\lambda^2}{2}T_a\Big)\Big] = 1. \tag{7}$$

Fix any $\mu > 0$. For $\lambda := \sqrt{2\mu}$, (7) yields the desired result.

b) For any $\lambda > 0$, consider the martingale $(N_t^\lambda)_{t\geq 0}$ defined by

$$N_t^\lambda := \frac{M_t^\lambda + M_t^{-\lambda}}{2} = \cosh\big(\lambda B_t\big)\exp\Big(-\frac{\lambda^2}{2}t\Big) = \cosh\big(\lambda|B_t|\big)\exp\Big(-\frac{\lambda^2}{2}t\Big).$$

The procedure in a) (using now N^λ instead of M^λ and $\overline{T}_a$ instead of T_a), using the inequality $0 \leq N_{\overline{T}_a\wedge n} \leq \cosh(\lambda a)$, yields

$$\cosh(\lambda a)\, E\Big[\exp\Big(-\frac{\lambda^2}{2}\overline{T}_a\Big)\mathbf{1}_{\{\overline{T}_a<\infty\}}\Big] = 1.$$

By definition, we see that $\overline{T}_a \leq T_a$. Thus, we deduce from a) that $P\big[\overline{T}_a < \infty\big] = 1$. Therefore,

$$\cosh(\lambda a)\, E\Big[\exp\Big(-\frac{\lambda^2}{2}\overline{T}_a\Big)\Big] = 1. \tag{8}$$

Fix any $\mu > 0$. For $\lambda := \sqrt{2\mu}$, (8) yields the desired result.